Bosnia and Herzegovina Team Selection Test 2005

www.artofproblemsolving.com/community/c537030

by gobathegreat

– Day 1

1 Let H be an orthocenter of an acute triangle ABC. Prove that midpoints of AB and CH and intersection point of angle bisectors of $\angle CAH$ and $\angle CBH$ lie on the same line.

2 If a_1, a_2 and a_3 are nonnegative real numbers for which $a_1+a_2+a_3=1$, then prove the inequality $a_1\sqrt{a_2}+a_2\sqrt{a_3}+a_3\sqrt{a_1}\leq \frac{1}{\sqrt{3}}$

3 Let n be a positive integer such that $n \geq 2$. Let $x_1, x_2,..., x_n$ be n distinct positive integers and S_i sum of all numbers between them except x_i for $i=1,2,...,n$. Let $f(x_1,x_2,...,x_n)=\frac{GCD(x_1,S_1)+GCD(x_2,S_2)+...+GCD(x_n,S_n)}{x_1+x_2+...+x_n}$.
Determine maximal value of $f(x_1,x_2,...,x_n)$, while $(x_1,x_2,...,x_n)$ is an element of set which consists from all n-tuples of distinct positive integers.

– Day 2

4 On the line which contains diameter PQ of circle $k(S,r)$, point A is chosen outside the circle such that tangent t from point A touches the circle in point T. Tangents on circle k in points P and Q are p and q, respectively. If $PT \cap q=N$ and $QT \cap p = M$, prove that points A, M and N are collinear.

5 If for an arbitrary permutation $(a_1,a_2,...,a_n)$ of set $1,2,...,n$ holds $\frac{{a_k}^2}{a_{k+1}}\leq k+2$, $k=1,2,...,n-1$, prove that $a_k=k$ for $k=1,2,...,n$

6 Let a, b and c are integers such that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=3$. Prove that abc is a perfect cube of an integer.