Iran Team Selection Test 2002

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1 $ABCD$ is a convex quadrilateral. We draw its diagnals to divide the quadrilateral to four triabgles. P is the intersection of diagnals. I_1, I_2, I_3, I_4 are excenters of PAD, PAB, PBC, PCD(excenters corresponding vertex P). Prove that I_1, I_2, I_3, I_4 lie on a circle iff $ABCD$ is a tangential quadrilateral.

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2 n people (with names $1, 2, \dots, n$) are around a table. Some of them are friends. At each step 2 friend can change their place. Find a necessary and sufficient condition for friendship relation between them that with these steps we can always reach to all of posiible permutations.

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3 A "*2-line*" is the area between two parallel lines. Length of "2-line" is distance of two parallel lines. We have covered unit circle with some "2-lines". Prove sum of lengths of "2-lines" is at least 2.

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4 O is a point in triangle ABC. We draw perpendicular from O to BC, AC, AB which intersect BC, AC, AB at A_1, B_1, C_1. Prove that O is circumcenter of triangle ABC iff perimeter of ABC is not less than perimeter of triangles $AB_1C_1, BC_1A_1, CB_1A_1$.

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5 A school has n students and k classes. Every two students in the same class are friends. For each two different classes, there are two people from these classes that are not friends. Prove that we can divide students into $n-k+1$ parts taht students in each part are not friends.

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6 Assume $x_1, x_2, \dots, x_n \in \mathbb{R}^+$, $\sum_{i=1}^n x_i^2 = n$, $\sum_{i=1}^n x_i \geq s > 0$ and $0 \leq \lambda \leq 1$. Prove that at least $\left\lceil \frac{s^2(1-\lambda)^2}{n} \right\rceil$ of these numbers are larger than $\frac{\lambda s}{n}$.

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7 S_1, S_2, S_3 are three spheres in $\mathbb{R}^3$ that their centers are not collinear. $k \leq 8$ is the number of planes that touch three spheres. A_i, B_i, C_i is the point that i-th plane touch the spheres S_1, S_2, S_3. Let O_i be circumcenter of $A_iB_iC_i$. Prove that O_i are collinear.

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8 We call A_1, A_2, A_3 *mangool* iff there is a permutation π that $A_{\pi(2)} \not\subset A_{\pi(1)}, A_{\pi(3)} \not\subset A_{\pi(1)} \cup A_{\pi(2)}$. A good family is a family of finite subsets of $\mathbb{N}$ like $X, A_1, A_2, \dots, A_n$. To each goo family we correspond a graph with vertices $\{A_1, A_2, \dots, A_n\}$. Connect A_i, A_j iff X, A_i, A_j are mangool sets. Find all graphs that we can find a good family corresponding to it.

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9 $\pi(n)$ is the number of primes that are not bigger than n. For $n = 2, 3, 4, 6, 8, 33, \dots$ we have $\pi(n)|n$. Does exist infinitely many integers n that $\pi(n)|n$?

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10 Suppose from $(m+2)\times(n+2)$ rectangle we cut 4, 1×1 corners. Now on first and last row first and last columns we write $2(m+n)$ real numbers. Prove we can fill the interior $m\times n$ rectangle with real numbers that every number is average of it's 4 neighbors.

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11 A $10\times 10\times 10$ cube has 1000 unit cubes. 500 of them are coloured black and 500 of them are coloured white. Show that there are at least 100 unit squares, being the common face of a white and a black unit cube.

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12 We call a permutation $(a_1, a_2, ..., a_n)$ of $(1, 2, ..., n)$ *quadratic* if there exists at least a perfect square among the numbers a_1, $a_1 + a_2$, ..., $a_1 + a_2 + ... + a_n$. Find all natural numbers n such that all permutations in S_n are quadratic.

Remark. S_n denotes the n-th symmetric group, the group of permutations on n elements.

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13 Let ABC be a triangle. The incircle of triangle ABC touches the side BC at A', and the line AA' meets the incircle again at a point P. Let the lines CP and BP meet the incircle of triangle ABC again at N and M, respectively. Prove that the lines AA', BN and CM are concurrent.
