Iran Team Selection Test 2006

www.artofproblemsolving.com/community/c5382

by Nima Ahmadi Pour, ZetaX, Ramtin Madani, grobber, Severius, Rust

Day 1

---

1 Suppose that p is a prime number.
Find all natural numbers n such that $p|\varphi(n)$ and for all a such that $(a,n)=1$ we have

$$n|a^{\frac{\varphi(n)}{p}}-1$$

---

2 Suppose n coins are available that their mass is unknown. We have a pair of balances and every time we can choose an even number of coins and put half of them on one side of the balance and put another half on the other side, therefore a *comparison* will be done. Our aim is determining that the mass of all coins is equal or not. Show that at least $n-1$ *comparisons* are required.

---

3 Suppose ABC is a triangle with M the midpoint of BC.
Suppose that AM intersects the incircle at K,L.
We draw parallel line from K and L to BC and name their second intersection point with incircle X and Y. Suppose that AX and AY intersect BC at P and Q.
Prove that $BP=CQ$.

Day 2

---

4 Let $x_1,x_2,\ldots,x_n$ be real numbers. Prove that

$$\sum_{i,j=1}^{n}|x_i+x_j|\geq n\sum_{i=1}^{n}|x_i|$$

---

5 Let ABC be a triangle such that it's circumcircle radius is equal to the radius of outer inscribed circle with respect to A.
Suppose that the outer inscribed circle with respect to A touches BC,AC,AB at M,N,L.
Prove that O (Center of circumcircle) is the orthocenter of MNL.

---

6 Let G be a tournoment such that it's edges are colored either red or blue.
Prove that there exists a vertex of G like v with the property that, for every other vertex u there is a mono-color directed path from v to u.

Day 3

1 We have n points in the plane, no three on a line.
We call k of them good if they form a convex polygon and there is no other point in the convex polygon.
Suppose that for a fixed k the number of k good points is c_k.
Show that the following sum is independent of the structure of points and only depends on n :

$$\sum_{i=3}^{n}(-1)^i c_i$$

2 Let n be a fixed natural number.
a) Find all solutions to the following equation :

$$\sum_{k=1}^{n}[\frac{x}{2^k}] = x-1$$

b) Find the number of solutions to the following equation (m is a fixed natural) :

$$\sum_{k=1}^{n}[\frac{x}{2^k}] = x-m$$

3 Let l, m be two parallel lines in the plane.
Let P be a fixed point between them.
Let E, F be variable points on l, m such that the angle EPF is fixed to a number like α where $0 < \alpha < \frac{\pi}{2}$.
(By angle EPF we mean the directed angle)
Show that there is another point (not P) such that it sees the segment EF with a fixed angle too.

Day 4

4 Let n be a fixed natural number.
Find all n tuples of natural pairwise distinct and coprime numbers like $a_1, a_2, \ldots, a_n$ such that for $1 \leq i \leq n$ we have

$$a_1 + a_2 + \ldots + a_n | a_1^i + a_2^i + \ldots + a_n^i$$

5 Let ABC be an acute angle triangle.
Suppose that D, E, F are the feet of perpendiclular lines from A, B, C to BC, CA, AB.

Let P,Q,R be the feet of perpendicular lines from A,B,C to EF,FD,DE.
Prove that

$$2(PQ+QR+RP)\geq DE+EF+FD$$

---

6 Suppose we have a simple polygon (that is it does not intersect itself, but not necessarily convex).
Show that this polygon has a diameter which is completely inside the polygon and the two arcs it creates on the polygon perimeter (the two arcs have 2 vertices in common) both have at least one third of the vertices of the polygon.
