Iran Team Selection Test 2008

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by Omid Hatami

Day 1

1 Find all functions $f:\mathbb{R}\longrightarrow\mathbb{R}$ such that for each $x,y\in\mathbb{R}$:

$$f(xf(y))+y+f(x)=f(x+f(y))+yf(x)$$

2 Suppose that I is incenter of triangle ABC and l' is a line tangent to the incircle. Let l be another line such that intersects AB,AC,BC respectively at C',B',A'. We draw a tangent from A' to the incircle other than BC, and this line intersects with l' at A_1. B_1,C_1 are similarly defined. Prove that AA_1,BB_1,CC_1 are concurrent.

3 Suppose that T is a tree with k edges. Prove that the k-dimensional cube can be partitioned to graphs isomorphic to T.

Day 2

4 Let P_1,P_2,P_3,P_4 be points on the unit sphere. Prove that $\sum_{i\neq j}\frac{1}{|P_i-P_j|}$ takes its minimum value if and only if these four points are vertices of a regular pyramid.

5 Let $a,b,c>0$ and $ab+bc+ca=1$. Prove that:

$$\sqrt{a^3+a}+\sqrt{b^3+b}+\sqrt{c^3+c}\geq 2\sqrt{a+b+c}.$$

6 Prove that in a tournament with 799 teams, there exist 14 teams, that can be partitioned into groups in a way that all of the teams in the first group have won all of the teams in the second group.

Day 3

7 Let S be a set with n elements, and F be a family of subsets of S with 2^{n-1} elements, such that for each $A,B,C\in F$, $A\cap B\cap C$ is not empty. Prove that the intersection of all of the elements of F is not empty.

8 Find all polynomials p of one variable with integer coefficients such that if a and b are natural numbers such that $a+b$ is a perfect square, then $p\left(a\right)+p\left(b\right)$ is also a perfect square.

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9 I_a is the excenter of the triangle ABC with respect to A, and AI_a intersects the circumcircle of ABC at T. Let X be a point on TI_a such that $XI_a^2=XA.XT$. Draw a perpendicular line from X to BC so that it intersects BC in A'. Define B' and C' in the same way. Prove that AA', BB' and CC' are concurrent.

Day 4

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10 In the triangle ABC, $\angle B$ is greater than $\angle C$. T is the midpoint of the arc BAC from the circumcircle of ABC and I is the incenter of ABC. E is a point such that $\angle AEI=90^\circ$ and $AE\parallel BC$. TE intersects the circumcircle of ABC for the second time in P. If $\angle B=\angle IPB$, find the angle $\angle A$.

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11 k is a given natural number. Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that for each $m,n\in\mathbb{N}$ the following holds:

$$f(m)+f(n)\mid(m+n)^k$$

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12 In the acute-angled triangle ABC, D is the intersection of the altitude passing through A with BC and I_a is the excenter of the triangle with respect to A. K is a point on the extension of AB from B, for which $\angle AKI_a=90^\circ+\frac{3}{4}\angle C$. I_aK intersects the extension of AD at L. Prove that DI_a bisects the angle $\angle AI_aB$ iff $AL=2R$. (R is the circumradius of ABC)
