Iran Team Selection Test 2009

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Day 1

1 Let ABC be a triangle and A' , B' and C' lie on BC , CA and AB respectively such that the incenter of $A'B'C'$ and ABC are coincide and the inradius of $A'B'C'$ is half of inradius of ABC . Prove that ABC is equilateral .

2 Let a be a fix natural number . Prove that the set of prime divisors of $2^{2^n} + a$ for $n = 1, 2, \cdots$ is infinite

3 Suppose that a,b,c be three positive real numbers such that $a + b + c = 3$. Prove that : $\frac{1}{2+a^2+b^2} + \frac{1}{2+b^2+c^2} + \frac{1}{2+c^2+a^2} \leq \frac{3}{4}$

Day 2

4 Find all polynomials f with integer coefficient such that, for every prime p and natural numbers u and v with the condition:

$$p \mid uv - 1$$

we always have $p \mid f(u)f(v) - 1$.

5 ABC is a triangle and AA' , BB' and CC' are three altitudes of this triangle . Let P be the feet of perpendicular from C' to $A'B'$, and Q is a point on $A'B'$ such that $QA = QB$. Prove that : $\angle PBQ = \angle PAQ = \angle PC'C$

6 We have a closed path on a vertices of a nn square which pass from each vertice exactly once . prove that we have two adjacent vertices such that if we cut the path from these points then length of each pieces is not less than quarter of total path .

Day 3

7 Suppose three direction on the plane . We draw 11 lines in each direction . Find maximum number of the points on the plane which are on three lines .

8 Find all polynomials $P(x, y)$ such that for all reals x and y,

$$P(x^2, y^2) = P\left(\frac{(x+y)^2}{2}, \frac{(x-y)^2}{2}\right).$$

9 In triangle ABC, D, E and F are the points of tangency of incircle with the center of I to BC, CA and AB respectively. Let M be the foot of the perpendicular from D to EF. P is on DM such that $DP = MP$. If H is the orthocenter of BIC, prove that PH bisects EF.

Day 4

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10 Let ABC be a triangle and $AB \neq AC$. D is a point on BC such that $BA = BD$ and B is between C and D . Let I_c be center of the circle which touches AB and the extensions of AC and BC . CI_c intersect the circumcircle of ABC again at T .
If $\angle TDI_c = \frac{\angle B + \angle C}{4}$ then find $\angle A$

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11 Let n be a positive integer. Prove that

$$3^{\dfrac{5^{2^n}-1}{2^{n+2}}} \equiv (-5)^{\dfrac{3^{2^n}-1}{2^{n+2}}} \pmod{2^{n+4}}.$$

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12 T is a subset of $1, 2, ..., n$ which has this property : for all distinct $i, j \in T$, $2j$ is not divisible by i . Prove that : $|T| \leq \frac{4}{9}n + \log_2 n + 2$
