Iran Team Selection Test 2010

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Day 1

1 Let $f:\mathbb{N}\to\mathbb{N}$ be a non-decreasing function and let n be an arbitrary natural number. Suppose that there are prime numbers $p_1,p_2,\dots,p_n$ and natural numbers $s_1,s_2,\dots,s_n$ such that for each $1\leq i\leq n$ the set $\{f(p_ir+s_i)|r=1,2,\dots\}$ is an infinite arithmetic progression. Prove that there is a natural number a such that

$$f(a+1),f(a+2),\dots,f(a+n)$$

form an arithmetic progression.

2 Find all non-decreasing functions $f:\mathbb{R}^+\cup\{0\}\to\mathbb{R}^+\cup\{0\}$ such that for each $x,y\in\mathbb{R}^+\cup\{0\}$

$$f\left(\frac{x+f(x)}{2}+y\right)=2x-f(x)+f(f(y)).$$

3 Find all two-variable polynomials $p(x,y)$ such that for each $a,b,c\in\mathbb{R}$:

$$p(ab,c^2+1)+p(bc,a^2+1)+p(ca,b^2+1)=0$$

Day 2

4 S,T are two trees without vertices of degree 2. To each edge is associated a positive number which is called length of this edge. Distance between two arbitrary vertices v,w in this graph is defined by sum of length of all edges in the path between v and w. Let f be a bijective function from leaves of S to leaves of T, such that for each two leaves u,v of S, distance of u,v in S is equal to distance of $f(u),f(v)$ in T. Prove that there is a bijective function g from vertices of S to vertices of T such that for each two vertices u,v of S, distance of u,v in S is equal to distance of $g(u)$ and $g(v)$ in T.

5 Circles W_1,W_2 intersect at P,K. XY is common tangent of two circles which is nearer to P and X is on W_1 and Y is on W_2. XP intersects W_2 for the second time in C and YP intersects W_1 in B. Let A be intersection point of BX and CY. Prove that if Q is the second intersection point of circumcircles of ABC and AXY

$$\angle QXA=\angle QKP$$

6 Let M be an arbitrary point on side BC of triangle ABC. W is a circle which is tangent to AB and BM at T and K and is tangent to circumcircle of AMC at P. Prove that if $TK||AM$, circumcircles of APT and KPC are tangent together.

Day 3

7 Without lifting pen from paper, we draw a polygon in such away that from every two adjacent sides one of them is vertical.
In addition, while drawing the polygon all vertical sides have been drawn from up to down. Prove that this polygon has cut itself.

8 Let ABC an isosceles triangle and $BC > AB = AC$. D, M are respectively midpoints of BC, AB. X is a point such that $BX \perp AC$ and $XD||AB$. BX and AD meet at H. If P is intersection point of DX and circumcircle of AHX (other than X), prove that tangent from A to circumcircle of triangle AMP is parallel to BC.

9 Sequence of real numbers $a_0, a_1, \dots, a_{1389}$ are called concave if for each $0 < i < 1389$, $a_i \geq \frac{a_{i-1}+a_{i+1}}{2}$. Find the largest c such that for every concave sequence of non-negative real numbers:

$$\sum_{i=0}^{1389} i a_i^2 \geq c \sum_{i=0}^{1389} a_i^2$$

Day 4

10 In every 1×1 square of an $m \times n$ table we have drawn one of two diagonals. Prove that there is a path including these diagonals either from left side to the right side, or from the upper side to the lower side.

11 Let O, H be circumcenter and orthogonal center of triangle ABC. M, N are midpoints of BH and CH. BB' is diagonal of circumcircle. If $HONM$ is a cyclic quadrilateral, prove that $B'N = \frac{1}{2}AC$.

12 Prove that for each natural number m, there is a natural number N such that for each b that $2 \leq b \leq 1389$ sum of digits of N in base b is larger than m.