Iran Team Selection Test 2011

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Day 1

1 In acute triangle ABC angle B is greater than$C$. Let M is midpoint of BC. D and E are the feet of the altitude from C and B respectively. K and L are midpoint of ME and MD respectively. If KL intersect the line through A parallel to BC in T, prove that $TA = TM$.

2 Find all natural numbers n greater than 2 such that there exist n natural numbers $a_1, a_2, \ldots, a_n$ such that they are not all equal, and the sequence $a_1a_2, a_2a_3, \ldots, a_na_1$ forms an arithmetic progression with nonzero common difference.

3 There are n points on a circle ($n > 1$). Define an "interval" as an arc of a circle such that it's start and finish are from those points. Consider a family of intervals F such that for every element of F like A there is almost one other element of F like B such that $A \subseteq B$ (in this case we call A is sub-interval of B). We call an interval maximal if it is not a sub-interval of any other interval. If m is the number of maximal elements of F and a is number of non-maximal elements of F, prove that $n \geq m + \frac{a}{2}$.

Day 2

4 Define a finite set A to be 'good' if it satisfies the following conditions:

-**(a)** For every three disjoint element of A, like a, b, c we have $\gcd(a, b, c) = 1$;

-**(b)** For every two distinct $b, c \in A$, there exists an $a \in A$, distinct from b, c such that bc is divisible by a.

Find all good sets.

5 Find all surjective functions $f : \mathbb{R} \to \mathbb{R}$ such that for every $x, y \in \mathbb{R}$, we have

$$f(x + f(x) + 2f(y)) = f(2x) + f(2y).$$

6 The circle ω with center O has given. From an arbitrary point T outside of ω draw tangents TB and TC to it. K and H are on TB and TC respectively.

a) B' and C' are the second intersection point of OB and OC with ω respectively. K' and H' are on angle bisectors of $\angle BCO$ and $\angle CBO$ respectively such that $KK' \perp BC$ and $HH' \perp BC$. Prove that K, H', B' are collinear if and only if H, K', C' are collinear.

b) Consider there exist two circle in TBC such that they are tangent two each other at J and both of them are tangent to ω.and one of them is tangent to TB at K and other one is tangent to TC at H. Prove that two quadrilateral $BKJI$ and $CHJI$ are cyclic (I is incenter of triangle OBC).

Day 3

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7 Find the locus of points P in an equilateral triangle ABC for which the square root of the distance of P to one of the sides is equal to the sum of the square root of the distance of P to the two other sides.

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8 Let p be a prime and k a positive integer such that $k \leq p$. We know that $f(x)$ is a polynomial in $\mathbb{Z}[x]$ such that for all $x \in \mathbb{Z}$ we have $p^k | f(x)$.

(a) Prove that there exist polynomials $A_0(x), \ldots, A_k(x)$ all in $\mathbb{Z}[x]$ such that

$$f(x) = \sum_{i=0}^{k} (x^p - x)^i p^{k-i} A_i(x),$$

(b) Find a counter example for each prime p and each $k > p$.

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9 We have n points in the plane such that they are not all collinear. We call a line ℓ a 'good' line if we can divide those n points in two sets A, B such that the sum of the distances of all points in A to ℓ is equal to the sum of the distances of all points in B to ℓ. Prove that there exist infinitely many points in the plane such that for each of them we have at least $n+1$ good lines passing through them.

Day 4

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10 Find the least value of k such that for all $a, b, c, d \in \mathbb{R}$ the inequality

$$\sqrt{(a^2+1)(b^2+1)(c^2+1)} + \sqrt{(b^2+1)(c^2+1)(d^2+1)} + \sqrt{(c^2+1)(d^2+1)(a^2+1)} + \sqrt{(d^2+1)(a^2+1)(b}$$

$$\geq 2(ab + bc + cd + da + ac + bd) - k$$

holds.

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11 Let ABC be a triangle and A', B', C' be the midpoints of BC, CA, AB respectively. Let P and P' be points in plane such that $PA = P'A', PB = P'B', PC = P'C'$. Prove that all PP' pass through a fixed point.

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12 Suppose that $f : \mathbb{N} \rightarrow \mathbb{N}$ is a function for which the expression $af(a) + bf(b) + 2ab$ for all $a, b \in \mathbb{N}$ is always a perfect square. Prove that $f(a) = a$ for all $a \in \mathbb{N}$.
