Iran Team Selection Test 2012

www.artofproblemsolving.com/community/c5388

by goldeneagle, goodar2006

– Exam 1

Day 1

1 Find all positive integers $n \geq 2$ such that for all integers i, j that $0 \leq i, j \leq n$, $i+j$ and $\binom{n}{i} + \binom{n}{j}$ have same parity.

Proposed by Mr.Etesami

2 Consider ω is circumcircle of an acute triangle ABC. D is midpoint of arc BAC and I is incenter of triangle ABC. Let DI intersect BC in E and ω for second time in F. Let P be a point on line AF such that PE is parallel to AI. Prove that PE is bisector of angle BPC.

Proposed by Mr.Etesami

3 Let n be a positive integer. Let S be a subset of points on the plane with these conditions:

$i)$ There does not exist n lines in the plane such that every element of S be on at least one of them.

$ii)$ for all $X \in S$ there exists n lines in the plane such that every element of $S - X$ be on at least one of them.

Find maximum of $\mid S \mid$.

Proposed by Erfan Salavati

Day 2

1 Consider $m+1$ horizontal and $n+1$ vertical lines ($m, n \geq 4$) in the plane forming an $m \times n$ table. Cosider a closed path on the segments of this table such that it does not intersect itself and also it passes through all $(m-1)(n-1)$ interior vertices (each vertex is an intersection point of two lines) and it doesn't pass through any of outer vertices. Suppose A is the number of vertices such that the path passes through them straight forward, B number of the table squares that only their two opposite sides are used in the path, and C number of the table squares that none of their sides is used in the path. Prove that

$$A = B - C + m + n - 1.$$

Proposed by Ali Khezeli

2 The function $f:\mathbb{R}^{\ge 0} \longrightarrow \mathbb{R}^{\ge 0}$ satisfies the following properties for all $a,b \in \mathbb{R}^{\ge 0}$:

a) $f(a)=0 \Leftrightarrow a=0$

b) $f(ab)=f(a)f(b)$

c) $f(a+b) \le 2\max\{f(a),f(b)\}$.

Prove that for all $a,b \in \mathbb{R}^{\ge 0}$ we have $f(a+b) \le f(a)+f(b)$.

Proposed by Masoud Shafaei

3 The pentagon $ABCDE$ is inscribed in a circle w. Suppose that w_a,w_b,w_c,w_d,w_e are reflections of w with respect to sides AB,BC,CD,DE,EA respectively. Let A' be the second intersection point of w_a,w_e and define B',C',D',E' similarly. Prove that

$$2 \le \frac{S_{A'B'C'D'E'}}{S_{ABCDE}} \le 3,$$

where S_X denotes the surface of figure X.

Proposed by Morteza Saghafian, Ali khezeli

– Exam 2

Day 1

1 Is it possible to put $\binom{n}{2}$ consecutive natural numbers on the edges of a complete graph with n vertices in a way that for every path (or cycle) of length 3 where the numbers a,b and c are written on its edges (edge b is between edges c and a), b is divisible by the greatest common divisor of the numbers a and c?

Proposed by Morteza Saghafian

2 Let $g(x)$ be a polynomial of degree at least 2 with all of its coefficients positive. Find all functions $f:\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$f(f(x)+g(x)+2y)=f(x)+g(x)+2f(y) \quad \forall x,y \in \mathbb{R}^+.$$

Proposed by Mohammad Jafari

3 Suppose $ABCD$ is a parallelogram. Consider circles w_1 and w_2 such that w_1 is tangent to segments AB and AD and w_2 is tangent to segments BC and CD. Suppose that there exists a circle which is tangent to lines AD and DC and externally tangent to w_1 and w_2. Prove that there exists a circle which is tangent to lines AB and BC and also externally tangent to circles w_1 and w_2.

Proposed by Ali Khezeli

Day 2

---

1 For positive reals a, b and c with $ab + bc + ca = 1$, show that

$$\sqrt{3}(\sqrt{a} + \sqrt{b} + \sqrt{c}) \le \frac{a\sqrt{a}}{bc} + \frac{b\sqrt{b}}{ca} + \frac{c\sqrt{c}}{ab}.$$

Proposed by Morteza Saghafian

---

2 Points A and B are on a circle ω with center O such that $\frac{\pi}{3} < \angle AOB < \frac{2\pi}{3}$. Let C be the circumcenter of the triangle AOB. Let l be a line passing through C such that the angle between l and the segment OC is $\frac{\pi}{3}$. l cuts tangents in A and B to ω in M and N respectively. Suppose circumcircles of triangles CAM and CBN, cut ω again in Q and R respectively and theirselves in P (other than C). Prove that $OP \perp QR$.

Proposed by Mehdi E'tesami Fard, Ali Khezeli

---

3 We call a subset B of natural numbers *loyal* if there exists natural numbers $i \le j$ such that $B = \{i, i+1, \dots, j\}$. Let Q be the set of all *loyal* sets. For every subset $A = \{a_1 < a_2 < \dots < a_k\}$ of $\{1, 2, \dots, n\}$ we set

$$f(A) = \max_{1 \le i \le k-1} a_{i+1} - a_i \quad \text{and} \quad g(A) = \max_{B \subseteq A, B \in Q} |B|.$$

Furthermore, we define

$$F(n) = \sum_{A \subseteq \{1,2,\dots,n\}} f(A) \quad \text{and} \quad G(n) = \sum_{A \subseteq \{1,2,\dots,n\}} g(A).$$

Prove that there exists $m \in \mathbb{N}$ such that for each natural number $n > m$ we have $F(n) > G(n)$. (By $|A|$ we mean the number of elements of A, and if $|A| \le 1$, we define $f(A)$ to be zero).

Proposed by Javad Abedi

– Exam 3

Day 1

---

1 Consider a regular 2^k-gon with center O and label its sides clockwise by $l_1, l_2, ..., l_{2^k}$. Reflect O with respect to l_1, then reflect the resulting point with respect to l_2 and do this process until the last side. Prove that the distance between the final point and O is less than the perimeter of the 2^k-gon.

Proposed by Hesam Rajabzade

---

2 Do there exist 2000 real numbers (not necessarily distinct) such that all of them are not zero and if we put any group containing 1000 of them as the roots of a monic polynomial of degree 1000, the coefficients of the resulting polynomial (except the coefficient of x^{1000}) be a permutation of the 1000 remaining numbers?

Proposed by Morteza Saghafian

---

3 Find all integer numbers x and y such that:

$$(y^3+xy-1)(x^2+x-y)=(x^3-xy+1)(y^2+x-y).$$

Proposed by Mahyar Sefidgaran

Day 2

---

1 Suppose p is an odd prime number. We call the polynomial $f(x)=\sum_{j=0}^n a_jx^j$ with integer coefficients i-remainder if $\sum_{p-1|j,j>0} a_j \equiv i \pmod{p}$. Prove that the set $\{f(0),f(1),...,f(p-1)\}$ is a complete residue system modulo p if and only if polynomials $f(x),(f(x))^2,...,(f(x))^{p-2}$ are 0-remainder and the polynomial $(f(x))^{p-1}$ is 1-remainder.

Proposed by Yahya Motevassel

---

2 Let n be a natural number. Suppose A and B are two sets, each containing n points in the plane, such that no three points of a set are collinear. Let $T(A)$ be the number of broken lines, each containing $n-1$ segments, and such that it doesn't intersect itself and its vertices are points of A. Define $T(B)$ similarly. If the points of B are vertices of a convex n-gon (are in *convex position*), but the points of A are not, prove that $T(B)<T(A)$.

Proposed by Ali Khezeli

---

3 Let O be the circumcenter of the acute triangle ABC. Suppose points A',B' and C' are on sides BC,CA and AB such that circumcircles of triangles $AB'C',BC'A'$ and $CA'B'$ pass through O. Let ℓ_a be the radical axis of the circle with center B' and radius $B'C$ and circle with center C' and radius $C'B$. Define ℓ_b and ℓ_c similarly. Prove that lines ℓ_a,ℓ_b and ℓ_c form a triangle such that it's orthocenter coincides with orthocenter of triangle ABC.

Proposed by Mehdi E'tesami Fard
