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Art of Problem Solving

Iran Team Selection Test 2013

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– TST 1

Day 1

1 In acute-angled triangle ABC, let H be the foot of perpendicular from A to BC and also suppose that J and I are excenters oposite to the side AH in triangles ABH and ACH. If P is the point that incircle touches BC, prove that I, J, P, H are concyclic.

2 Find the maximum number of subsets from $\{1, ..., n\}$ such that for any two of them like A, B if $A \subset B$ then $|B - A| \geq 3$. (Here $|X|$ is the number of elements of the set X.)

3 For nonnegative integers m and n, define the sequence $a(m, n)$ of real numbers as follows. Set $a(0, 0) = 2$ and for every natural number n, set $a(0, n) = 1$ and $a(n, 0) = 2$. Then for $m, n \geq 1$, define

$$a(m, n) = a(m - 1, n) + a(m, n - 1).$$

Prove that for every natural number k, all the roots of the polynomial $P_k(x) = \sum_{i=0}^{k} a(i, 2k + 1 - 2i)x^i$ are real.

Day 2

4 m and n are two nonnegative integers. In the Philosopher's Chess, The chessboard is an infinite grid of identical regular hexagons and a new piece named the Donkey moves on it as follows:

Starting from one of the hexagons, the Donkey moves m cells in one of the 6 directions, then it turns 60 degrees clockwise and after that moves n cells in this new direction until it reaches it's final cell.

At most how many cells are in the Philosopher's chessboard such that one cannot go from anyone of them to the other with a finite number of movements of the Donkey?

Proposed by Shayan Dashmiz

5 Do there exist natural numbers a, b and c such that $a^2 + b^2 + c^2$ is divisible by $2013(ab + bc + ca)$?

Proposed by Mahan Malihi

6 Points A, B, C and D lie on line l in this order. Two circular arcs C_1 and C_2, which both lie on one side of line l, pass through points A and B and two circular arcs C_3 and C_4 pass through points C and D such that C_1 is tangent to C_3 and C_2 is tangent to C_4. Prove that the common external tangent of C_2 and C_3 and the common external tangent of C_1 and C_4 meet each other on line l.

Proposed by Ali Khezeli

– TST 2

Day 1

7 Nonnegative real numbers $p_1, \dots, p_n$ and $q_1, \dots, q_n$ are such that $p_1 + \dots + p_n = q_1 + \dots + q_n$ Among all the matrices with nonnegative entries having p_i as sum of the i-th row's entries and q_j as sum of the j-th column's entries, find the maximum sum of the entries on the main diagonal.

8 Find all Arithmetic progressions $a_1, a_2, ...$ of natural numbers for which there exists natural number $N > 1$ such that for every $k \in \mathbb{N}$:

$$a_1 a_2 ... a_k \mid a_{N+1} a_{N+2} ... a_{N+k}$$

9 find all functions $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ such that f is increasing and also:

$$f(f(x) + 2g(x) + 3f(y)) = g(x) + 2f(x) + 3g(y)$$

$$g(f(x) + y + g(y)) = 2x - g(x) + f(y) + y$$

Day 2

10 On each edge of a graph is written a real number,such that for every even tour of this graph,sum the edges with signs alternatively positive and negative is zero.prove that one can assign to each of the vertices of the graph a real number such that sum of the numbers on two adjacent vertices is the number on the edge between them.(tour is a closed path from the edges of the graph that may have repeated edges or vertices)

11 Let a, b, c be sides of a triangle such that $a \geq b \geq c$. prove that:

$$\sqrt{a(a+b-\sqrt{ab})} + \sqrt{b(a+c-\sqrt{ac})} + \sqrt{c(b+c-\sqrt{bc})} \geq a+b+c$$

12 Let $ABCD$ be a cyclic quadrilateral that inscribed in the circle ω.Let I_1, I_2 and r_1, r_2 be incenters and radii of incircles of triangles ACD and ABC,respectively.assume that $r_1 = r_2$. let ω' be a circle that touches AB, AD and touches ω at T. tangents from A, T to ω meet at the point K.prove that I_1, I_2, K lie on a line.

– TST 3

Day 1

13 P is an arbitrary point inside acute triangle ABC. Let A_1, B_1, C_1 be the reflections of point P with respect to sides BC, CA, AB. Prove that the centroid of triangle $A_1B_1C_1$ lies inside triangle ABC.

14 we are given n rectangles in the plane. Prove that between $4n$ right angles formed by these rectangles there are at least $[4\sqrt{n}]$ distinct right angles.

15 a) Does there exist a sequence $a_1 < a_2 < \dots$ of positive integers, such that there is a positive integer N that $\forall m > N$, a_m has exactly $d(m) - 1$ divisors among a_is?

b) Does there exist a sequence $a_1 < a_2 < \dots$ of positive integers, such that there is a positive integer N that $\forall m > N$, a_m has exactly $d(m) + 1$ divisors among a_is?

Day 2

16 The function $f : \mathbb{Z} \to \mathbb{Z}$ has the property that for all integers m and n

$$f(m) + f(n) + f(f(m^2 + n^2)) = 1.$$

We know that integers a and b exist such that $f(a) - f(b) = 3$. Prove that integers c and d can be found such that $f(c) - f(d) = 1$.

Proposed by Amirhossein Gorzi

17 In triangle ABC, AD and AH are the angle bisector and the altitude of vertex A, respectively. The perpendicular bisector of AD, intersects the semicircles with diameters AB and AC which are drawn outside triangle ABC in X and Y, respectively. Prove that the quadrilateral $XYDH$ is concyclic.

Proposed by Mahan Malihi

18 A special kind of parallelogram tile is made up by attaching the legs of two right isosceles triangles of side length 1. We want to put a number of these tiles on the floor of an $n \times n$ room such that the distance from each vertex of each tile to the sides of the room is an integer and also no two tiles overlap. Prove that at least an area n of the room will not be covered by the tiles.

Proposed by Ali Khezeli