Iran Team Selection Test 2014

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– TST 1

Day 1

1 suppose that O is the circumcenter of acute triangle ABC.
we have circle with center O that is tangent too BC that named w
suppose that X and Y are the points of intersection of the tangent from A to w with line BC(X and B are in the same side of AO) T is the intersection of the line tangent to circumcirle of ABC in B and the line from X parallel to AC. S is the intersection of the line tangent to circumcirle of ABC in C and the line from Y parallel to AB.
prove that ST is tangent ABC.

2 find all polynomials with integer coefficients that $P(\mathbb{Z})=p(a):a\in\mathbb{Z}$ has a Geometric progression.

3 we named a $n*n$ table $selfish$ if we number the row and column with $0,1,2,3,...,n-1$.(from left to right an from up to down)
for every $i,j\in{0,1,2,...,n-1}$ the number of cell (i,j) is equal to the number of number i in the row j.
for example we have such table for $n=5$
1 0 3 3 4
1 3 2 1 1
0 1 0 1 0
2 1 0 0 0
1 0 0 0 0
prove that for $n>5$ there is no $selfish$ table

Day 2

4 Find the maximum number of Permutation of set $1,2,3,...,2014$ such that for every 2 different number a and b in this set at last in one of the permutation b comes exactly after a

5 n is a natural number. for every positive real numbers $x_{1},x_{2},...,x_{n+1}$ such that $x_{1}x_{2}...x_{n+1}=1$ prove that: $\sqrt[x_{1}]{n}+...+\sqrt[x_{n+1}]{n} \geq n^{\sqrt[n]{x_{1}}}+...+n^{\sqrt[n]{x_{n+1}}}$

6 I is the incenter of triangle ABC. perpendicular from I to AI meet AB and AC at B' and C' respectively .

Suppose that B'' and C'' are points on half-line BC and CB such that $BB'' = BA$ and $CC'' = CA$.
Suppose that the second intersection of circumcircles of $AB'B''$ and $AC'C''$ is T.
Prove that the circumcenter of AIT is on the BC.

– TST 2

Day 1

1 Consider a tree with n vertices, labeled with $1, \dots, n$ in a way that no label is used twice. We change the labeling in the following way - each time we pick an edge that hasn't been picked before and swap the labels of its endpoints. After performing this action $n-1$ times, we get another tree with its labeling a permutation of the first graph's labeling.
Prove that this permutation contains exactly one cycle.

2 Point D is an arbitary point on side BC of triangle ABC. I,I_1 andI_2 are the incenters of triangles ABC,ABD and ACD respectively. $M \neq A$ and $N \neq A$ are the intersections of circumcircle of triangle ABC and circumcircles of triangles IAI_1 and IAI_2 respectively. Prove that regardless of point D, line MN goes through a fixed point.

3 prove for all $k > 1$ equation $(x+1)(x+2)...(x+k) = y^2$ has finite solutions.

Day 2

4 n is a natural number. We shall call a permutation $a_1, \dots, a_n$ of $1, \dots, n$ a quadratic(cubic) permutation if $\forall 1 \leq i \leq n-1$ we have $a_i a_{i+1} + 1$ is a perfect square(cube). (a) Prove that for infinitely many natural numbers n there exists a quadratic permutation. (b) Prove that for no natural number n exists a cubic permutation.

5 if $x, y, z > 0$ are postive real numbers such that $x^2 + y^2 + z^2 = x^2y^2 + y^2z^2 + z^2x^2$
prove that

$$((x-y)(y-z)(z-x))^2 \leq 2((x^2-y^2)^2 + (y^2-z^2)^2 + (z^2-x^2)^2)$$

6 Consider n segments in the plane which no two intersect and between their $2n$ endpoints no three are collinear. Is the following statement true?
Statement: There exists a simple $2n$-gon such that it's vertices are the $2n$ endpoints of the segments and each segment is either completely inside the polygon or an edge of the polygon.

– TST 3

Day 1

1 The incircle of a non-isosceles triangle ABC with the center I touches the sides BC, AC, AB at A_1, B_1, C_1 .
let AI, BI, CI meets BC, AC, AB at A_2, B_2, C_2.
let A' is a point on AI such that $A_1A' \perp B_2C_2$ .B', C' respectively.
prove that two triangle $A'B'C', A_1B_1C_1$ are equal.

2 is there a function $f : \mathbb{N} \to \mathbb{N}$ such that $i) \exists n \in \mathbb{N} : f(n) \neq n$ $ii)$ the number of divisors of m is $f(n)$ if and only if the number of divisors of $f(m)$ is n

3 let $m, n \in \mathbb{N}$ and $p(x), q(x), h(x)$ are polynomials with real Coefficients such that $p(x)$ is Descending.
and for all $x \in \mathbb{R}$ $p(q(nx + m) + h(x)) = n(q(p(x)) + h(x)) + m$.
prove that dont exist function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ $f(q(p(x)) + h(x)) = f(x)^2 + 1$

Day 2

4 Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $x, y \in \mathbb{R}^+$,

$$f\left(\frac{y}{f(x+1)}\right) + f\left(\frac{x+1}{xf(y)}\right) = f(y)$$

5 Given a set $X = \{x_1, \ldots, x_n\}$ of natural numbers in which for all $1 < i \leq n$ we have $1 \leq x_i - x_{i-1} \leq 2$, call a real number a **good** if there exists $1 \leq j \leq n$ such that $2|x_j - a| \leq 1$. Also a subset of X is called **compact** if the average of its elements is a good number.
Prove that at least 2^{n-3} subsets of X are compact.

Proposed by Mahyar Sefidgaran

6 The incircle of a non-isosceles triangle ABC with the center I touches the sides BC at D.
let X is a point on arc BC from circumcircle of triangle ABC such that if E, F are feet of perpendicular from X on BI, CI and M is midpoint of EF we have $MB = MC$.
prove that $\widehat{BAD} = \widehat{CAX}$