Australia National Olympiad 2006

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by Arne

Day 1

1	Find all positive integers m and n such that $1+5\cdot 2^m=n^2$.
2	Let f be a function defined on the positive integers, taking positive integral values, such that $f(a)f(b)=f(ab)$ for all positive integers a and b, $f(a)<f(b)$ if $a<b$, $f(3)\geq 7$. Find the smallest possible value of $f(3)$.
3	Let $PRUS$ be a trapezium such that $\angle PSR=2\angle QSU$ and $\angle SPU=2\angle UPR$. Let Q and T be on PR and SU respectively such that SQ and PU bisect $\angle PSR$ and $\angle SPU$ respectively. Let PT meet SQ at E. The line through E parallel to SR meets PU in F and the line through E parallel to PU meets SR in G. Let FG meet PR and SU in K and L respectively. Prove that KF = FG = GL.
4	There are n points on a circle, such that each line segment connecting two points is either red or blue. P_iP_j is red if and only if $P_{i+1}P_{j+1}$ is blue, for all distinct i,j in $\{1,2,...,n\}$. (a) For which values of n is this possible? (b) Show that one can get from any point on the circle to any other point, by doing a maximum of 3 steps, where one step is moving from a point to another point through a red segment connecting these points.

Day 2

1	In a square $ABCD$, E is a point on diagonal BD. P and Q are the circumcentres of $\triangle ABE$ and $\triangle ADE$ respectively. Prove that $APEQ$ is a square.
2	For any positive integer n, define a_n to be the product of the digits of n. (a) Prove that $n\geq a(n)$ for all positive integers n. (b) Find all n for which $n^2-17n+56=a(n)$.