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– Algebra

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1 Let Q be a polynomial

$$Q(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

where $a_0, \ldots, a_n$ are nonnegative integers. Given that $Q(1) = 4$ and $Q(5) = 152$, find $Q(6)$.

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2 The fraction $\frac{1}{2015}$ has a unique "(restricted) partial fraction decomposition" of the form

$$\frac{1}{2015} = \frac{a}{5} + \frac{b}{13} + \frac{c}{31},$$

where a, b, and c are integers with $0 \leq a < 5$ and $0 \leq b < 13$. Find $a + b$.

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3 Let p be a real number and $c \neq 0$ such that

$$c - 0.1 < x^p \left(\frac{1 - (1+x)^{10}}{1 + (1+x)^{10}} \right) < c + 0.1$$

for all (positive) real numbers x with $0 < x < 10^{-100}$. (The exact value 10^{-100} is not important. You could replace it with any "sufficiently small number".)

Find the ordered pair (p, c).

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4 Compute the number of sequences of integers $(a_1, \ldots, a_{200})$ such that the following conditions hold.

- $0 \leq a_1 < a_2 < \cdots < a_{200} \leq 202$.
- There exists a positive integer N with the following property: for every index $i \in \{1, \ldots, 200\}$ there exists an index $j \in \{1, \ldots, 200\}$ such that $a_i + a_j - N$ is divisible by 203.

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5 Let a, b, c be positive real numbers such that $a + b + c = 10$ and $ab + bc + ca = 25$. Let $m = \min\{ab, bc, ca\}$. Find the largest possible value of m.

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6 Let a, b, c, d, e be nonnegative integers such that $625a + 250b + 100c + 40d + 16e = 15^3$. What is the maximum possible value of $a + b + c + d + e$?

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7 Suppose (a_1, a_2, a_3, a_4) is a 4-term sequence of real numbers satisfying the following two conditions:

- $a_3 = a_2 + a_1$ and $a_4 = a_3 + a_2$;
- there exist real numbers a, b, c such that

$$an^2 + bn + c = \cos(a_n)$$

for all $n \in \{1, 2, 3, 4\}$.

Compute the maximum possible value of

$$\cos(a_1) - \cos(a_4)$$

over all such sequences (a_1, a_2, a_3, a_4).

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8 Find the number of ordered pairs of integers $(a, b) \in \{1, 2, \ldots, 35\}^2$ (not necessarily distinct) such that $ax + b$ is a "quadratic residue modulo $x^2 + 1$ and 35", i.e. there exists a polynomial $f(x)$ with integer coefficients such that either of the following *equivalent* conditions holds:

- there exist polynomials P, Q with integer coefficients such that $f(x)^2 - (ax + b) = (x^2 + 1)P(x) + 35Q(x)$;
- or more conceptually, the remainder when (the polynomial) $f(x)^2 - (ax + b)$ is divided by (the polynomial) $x^2 + 1$ is a polynomial with integer coefficients all divisible by 35.

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9 Let $N = 30^{2015}$. Find the number of ordered 4-tuples of integers $(A, B, C, D) \in \{1, 2, \ldots, N\}^4$ (not necessarily distinct) such that for every integer n, $An^3 + Bn^2 + 2Cn + D$ is divisible by N.

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10 Find all ordered 4-tuples of integers (a, b, c, d) (not necessarily distinct) satisfying the following system of equations:

$$\begin{aligned} a^2 - b^2 - c^2 - d^2 &= c - b - 2 \\ 2ab &= a - d - 32 \\ 2ac &= 28 - a - d \\ 2ad &= b + c + 31. \end{aligned}$$

– Geometry

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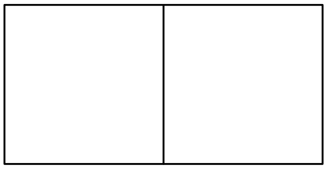
1 Let R be the rectangle in the Cartesian plane with vertices at $(0, 0)$, $(2, 0)$, $(2, 1)$, and $(0, 1)$. R can be divided into two unit squares, as shown.



Pro selects a point P at random in the interior of R. Find the probability that the line through P with slope $\frac{1}{2}$ will pass through both unit squares.

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2 Let ABC be a triangle with orthocenter H; suppose $AB = 13$, $BC = 14$, $CA = 15$. Let G_A be the centroid of triangle HBC, and define G_B, G_C similarly. Determine the area of triangle $G_AG_BG_C$.

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3 Let $ABCD$ be a quadrilateral with $\angle BAD = \angle ABC = 90^\circ$, and suppose $AB = BC = 1$, $AD = 2$. The circumcircle of ABC meets $\overline{AD}$ and $\overline{BD}$ at point E and F, respectively. If lines AF and CD meet at K, compute EK.

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4 Let $ABCD$ be a cyclic quadrilateral with $AB = 3$, $BC = 2$, $CD = 2$, $DA = 4$. Let lines perpendicular to $\overline{BC}$ from B and C meet $\overline{AD}$ at B' and C', respectively. Let lines perpendicular to $\overline{BC}$ from A and D meet $\overline{AD}$ at A' and D', respectively. Compute the ratio $\frac{[BCC'B']}{[DAA'D']}$, where $[\varpi]$ denotes the area of figure ϖ.

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5 Let I be the set of points (x, y) in the Cartesian plane such that

$$x > \left(\frac{y^4}{9} + 2015\right)^{1/4}$$

Let $f(r)$ denote the area of the intersection of I and the disk $x^2 + y^2 \le r^2$ of radius $r > 0$ centered at the origin $(0, 0)$. Determine the minimum possible real number L such that $f(r) < Lr^2$ for all $r > 0$.

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6 In triangle ABC, $AB = 2$, $AC = 1 + \sqrt{5}$, and $\angle CAB = 54^\circ$. Suppose D lies on the extension of AC through C such that $CD = \sqrt{5} - 1$. If M is the midpoint of BD, determine the measure of $\angle ACM$, in degrees.

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7 Let $ABCD$ be a square pyramid of height $\frac{1}{2}$ with square base $ABCD$ of side length $AB = 12$ (so E is the vertex of the pyramid, and the foot of the altitude from E to $ABCD$ is the center of square $ABCD$). The faces ADE and CDE meet at an acute angle of measure α (so that $0^\circ < \alpha < 90^\circ$). Find $\tan \alpha$.

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8 Let S be the set of **discs** D contained completely in the set $\{(x, y) : y < 0\}$ (the region below the x-axis) and centered (at some point) on the curve $y = x^2 - \frac{3}{4}$. What is the area of the union of the elements of S?

9 Let $ABCD$ be a regular tetrahedron with side length 1. Let X be the point in the triangle BCD such that $[XBC] = 2[XBD] = 4[XCD]$, where $[\varpi]$ denotes the area of figure ϖ. Let Y lie on segment AX such that $2AY = YX$. Let M be the midpoint of BD. Let Z be a point on segment AM such that the lines YZ and BC intersect at some point. Find $\frac{AZ}{ZM}$.

10 Let $\mathcal{G}$ be the set of all points (x, y) in the Cartesian plane such that $0 \le y \le 8$ and

$$(x-3)^2 + 31 = (y-4)^2 + 8\sqrt{y(8-y)}.$$

There exists a unique line ℓ of **negative slope** tangent to $\mathcal{G}$ and passing through the point $(0, 4)$. Suppose ℓ is tangent to $\mathcal{G}$ at a **unique** point P. Find the coordinates (α, β) of P.