ITAMO 2004

www.artofproblemsolving.com/community/c5394

by Sayan

1 Observing the temperatures recorded in Cesenatico during the December and January, Stefano noticed an interesting coincidence: in each day of this period, the low temperature is equal to the sum of the low temperatures the preceeding day and the succeeding day.
Given that the low temperatures in December 3 and January 31 were 5°C and 2°C respectively, find the low temperature in December 25.

---

2 Two parallel lines r, s and two points $P \in r$ and $Q \in s$ are given in a plane. Consider all pairs of circles (C_P, C_Q) in that plane such that C_P touches r at P and C_Q touches s at Q and which touch each other externally at some point T. Find the locus of T.

---

3 (a) Is 2005^{2004} the sum of two perfect squares?
(b) Is 2004^{2005} the sum of two perfect squares?

---

4 Antonio and Bernardo play the following game. They are given two piles of chips, one with m and the other with n chips. Antonio starts, and thereafter they make in turn one of the following moves:
(i) take a chip from one pile;
(ii) take a chip from each of the piles;
(ii) remove a chip from one of the piles and put it onto the other.
Who cannot make any more moves, loses. Decide, as a function of m and n if one of the players has a winning strategy, and in the case of the affirmative answer describe that strategy.

---

5 Decide if the following statement is true or false:
For every sequence $\{x_n\}_{n\in\mathbb{N}}$ of non-negative real numbers, there exist sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ of non-negative real numbers such that:
(a) $x_n = a_n + b_n$ for all n;
(b) $a_1 + \cdots + a_n \le n$ for infinitely many values of n;
(c) $b_1 + \cdots + b_n \le n$ for infinitely many values of n.

---

6 Let P be a point inside a triangle ABC. Lines AP, BP, CP meet the opposite sides of the triangle at points A', B', C' respectively. Denote $x = \frac{AP}{PA'}, y = \frac{BP}{PB'}$ and $z = \frac{CP}{PC'}$. Prove that $xyz = x + y + z + 2$.
