ITAMO 2005

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Day 1

1 Let ABC be a right angled triangle with hypotenuse AC, and let H be the foot of the altitude from B to AC. Knowing that there is a right-angled triangle with side-lengths AB, BC, BH, determine all the possible values of $\frac{AH}{CH}$.

2 Prove that among any 18 consecutive positive integers not exceeding 2005 there is at least one divisible by the sum of its digits.

3 In each cell of a 4×4 table a digit 1 or 2 is written. Suppose that the sum of the digits in each of the four 3×3 sub-tables is divisible by 4, but the sum of the digits in the entire table is not divisible by 4. Find the greatest and the smallest possible value of the sum of the 16 digits.

Day 2

1 Determine all $n \geq 3$ for which there are n positive integers $a_1, \cdots, a_n$ any two of which have a common divisor greater than 1, but any three of which are coprime. Assuming that, moreover, the numbers a_i are less than 5000, find the greatest possible n.

2 Let h be a positive integer. The sequence a_n is defined by $a_0 = 1$ and

$$a_{n+1} = \{ \begin{array}{l} \frac{a_n}{2} \text{ if } a_n \text{ is even} \\ a_n + h \text{ otherwise .} \end{array}$$

For example, $h = 27$ yields $a_1 = 28, a_2 = 14, a_3 = 7, a_4 = 34$ etc. For which h is there an $n > 0$ with $a_n = 1$?

3 Two circles γ_1, γ_2 in a plane, with centers A and B respectively, intersect at C and D. Suppose that the circumcircle of ABC intersects γ_1 in E and γ_2 in F, where the arc EF not containing C lies outside γ_1 and γ_2. Prove that this arc EF is bisected by the line CD.