ITAMO 2007

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by edriv

Day 1

1 It is given a regular hexagon in the plane. Let P be a point of the plane. Define s(P) as the sum of the distances from P to each side of the hexagon, and v(P) as the sum of the distances from P to each vertex.
a) Find the locus of points P that minimize s(P)
b) Find the locus of points P that minimize v(P)

2 We define two polynomials with integer coefficients P,Q to be similar if the coefficients of P are a permutation of the coefficents of Q.
a) if P,Q are similar, then $P(2007)-Q(2007)$ is even
b) does there exist an integer $k>2$ such that $k \mid P(2007)-Q(2007)$ for all similar polynomials P,Q?

3 Let ABC be a triangle, G its centroid, M the midpoint of AB, D the point on the line AG such that $AG=GD, A \neq D$, E the point on the line BG such that $BG=GE, B \neq E$. Show that the quadrilateral BDCM is cyclic if and only if $AD=BE$.

Day 2

4 Today is Barbara's birthday, and Alberto wants to give her a gift playing the following game. The numbers 0,1,2,...,1024 are written on a blackboard. First Barbara erases 2^9 numbers, then Alberto erases 2^8 numbers, then Barbara 2^7 and so on, until there are only two numbers a,b left. Now Barbara earns $|a-b|$ euro.
Find the maximum number of euro that Barbara can always win, independently of Alberto's strategy.

5 The sequence of integers $(a_n)_{n\geq 1}$ is defined by $a_1=2$, $a_{n+1}=2a_n^2-1$.
Prove that for each positive integer n, n and a_n are coprime.

6 a) For each $n \geq 2$, find the maximum constant c_n such that $\frac{1}{a_1+1}+\frac{1}{a_2+1}+\ldots+\frac{1}{a_n+1} \geq c_n$ for all positive reals $a_1,a_2,\ldots,a_n$ such that $a_1a_2\cdots a_n=1$.

b) For each $n \geq 2$, find the maximum constant d_n such that $\frac{1}{2a_1+1}+\frac{1}{2a_2+1}+\ldots+\frac{1}{2a_n+1} \geq d_n$ for all positive reals $a_1,a_2,\ldots,a_n$ such that $a_1a_2\cdots a_n=1$.