ITAMO 2009

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by Sayan

Day 1

1 Let $a < b < c < d < e$ be real numbers. We calculate all possible sums in pairs of these 5 numbers. Of these 10 sums, the three smaller ones are 32, 36, 37, while the two larger ones are 48 and 51. Determine all possible values that e can take.

2 $ABCD$ is a square with centre O. Two congruent isosceles triangle BCJ and CDK with base BC and CD respectively are constructed outside the square. let M be the midpoint of CJ. Show that OM and BK are perpendicular to each other.

3 A natural number n is called *nice* if it enjoys the following properties:
The expression is made up of 4 decimal digits;
the first and third digits of n are equal;
the second and fourth digits of n are equal;
the product of the digits of n divides n^2.
Determine all nice numbers.

Day 2

1 A flea is initially at the point $(0,0)$ in the Cartesian plane. Then it makes n jumps. The direction of the jump is taken in a choice of the four cardinal directions. The first step is of length 1, the second of length 2, the third of length 4, and so on. The n^{th}-jump is of length 2^{n-1}. Prove that, if you know the final position flea, then it is possible to uniquely determine its position after each of the n jumps.

2 Let ABC be an acute-angled scalene triangle and Γ be its circumcircle. K is the foot of the internal bisector of $\angle BAC$ on BC. Let M be the midpoint of the arc BC containing A. MK intersect Γ again at A'. T is the intersection of the tangents at A and A'. R is the intersection of the perpendicular to AK at A and perpendicular to $A'K$ at A'. Show that T, R and K are collinear.

3 A natural number k is said n-squared if by colouring the squares of a $2n \times k$ chessboard, in any manner, with n different colours, we can find 4 separate unit squares of the same colour, the centers of which are vertices of a rectangle having sides parallel to the sides of the board. Determine, in function of n, the smallest natural k that is n-squared.