ITAMO 2010

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1 In a mathematics test number of participants is $N < 40$. The passmark is fixed at 65. The test results are
the following:
The average of all participants is 66, that of the promoted 71 and that of the repeaters 56.
However, due to an error in the wording of a question, all scores are increased by 5. At this point
the average of the promoted participants becomes 75 and that of the non-promoted 59.
(a) Find all possible values of N.
(b) Find all possible values of N in the case where, after the increase, the average of the promoted had become 79 and that of non-promoted 47.

2 Every non-negative integer is coloured white or red, so that:
there are at least a white number and a red number;
the sum of a white number and a red number is white;
the product of a white number and a red number is red.
Prove that the product of two red numbers is always a red number, and the sum of two red numbers is always a red number.

3 Let $ABCD$ be a convex quadrilateral. such that $\angle CAB = \angle CDA$ and $\angle BCA = \angle ACD$. If M be the midpoint of AB, prove that $\angle BCM = \angle DBA$.

4 In a trapezium $ABCD$, the sides AB and CD are parallel and the angles $\angle ABC$ and $\angle BAD$ are acute. Show that it is possible to divide the triangle ABC into 4 disjoint triangle $X_1..., X_4$ and the triangle ABD into 4 disjoint triangles $Y_1, ..., Y_4$ such that the triangles X_i and Y_i are congruent for all i.

5 In the land of Cockaigne, people play the following solitaire. It starts from a finite string of zeros and ones, and are granted the following moves:
(i) cancel each two consecutive ones;
(ii) delete three consecutive zeros;
(iii) if the substring within the string is 01, one may replace this by substring 100.
The moves (i), (ii) and (iii) must be made one at a time. You win if you can reduce the string to a string formed by two digits or less.
(For example, starting from 0101, one can win using move (iii) first in the last two digits, resulting in 01100, then playing the move (i) on two 'ones', and finally the move (ii) on the three zeros,

one will get the empty string.)
Among all the 1024 possible strings of ten-digit binary numbers, how many are there from which it is not possible to win the solitary?

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6 Prove that there are infinitely many prime numbers that divide at least one integer of the form $2^{n^3+1} - 3^{n^2+1} + 5^{n+1}$ where n is a positive integer.
