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1 A trapezium is given with parallel bases having lengths 1 and 4. Split it into two trapeziums by a cut, parallel to the bases, of length 3. We now want to divide the two new trapeziums, always by means of cuts parallel to the bases, in m and n trapeziums, respectively, so that all the $m+n$ trapezoids obtained have the same area. Determine the minimum possible value for $m+n$ and the lengths of the cuts to be made to achieve this minimum value.

2 A sequence of positive integers $a_1, a_2, \ldots, a_n$ is called *ladder* of length n if it consists of n consecutive integers in ascending order.
(a) Prove that for every positive integer n there exist two ladders of length n, with no elements in common, $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, such that for all i between 1 and n, the greatest common divisor of a_i and b_i is equal to 1.
(b) Prove that for every positive integer n there exist two ladders of length n, with no elements in common, $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, such that for all i between 1 and n, the greatest common divisor of a_i and b_i is greater than 1.

3 Integers between 1 and 7 are written on a blackboard. It is possible that not all the numbers from 1 to 7 are present, and it is also possible that one, some or all of the numbers are repeated, one or more times.
A move consists of choosing one or more numbers on the blackboard, where all distinct, delete them and write different numbers in their place, such that the written numbers together with those deleted form the whole set $\{1, 2, 3, 4, 5, 6, 7\}$

For example, moves allowed are:
delete a 4 and a 5, and write in their place the numbers $1, 2, 3, 6$ and 7;
deleting a 1, a 2, a 3, a 4, a 5, a 6 and a 7 and write nothing in their place.

Prove that, if it is possible to find a sequence of moves, starting from the initial situation, leading to have on board a single number (written once), then this number does not depend on the sequence of moves used.

4 $ABCD$ is a convex quadrilateral. P is the intersection of external bisectors of $\angle DAC$ and $\angle DBC$. Prove that $\angle APD = \angle BPC$ if and only if $AD + AC = BC + BD$

5 Determine all solutions (p, n) of the equation

$$n^3 = p^2 - p - 1$$

where p is a prime number and n is an integer

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6 Let $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$. We want to color, using k colors, all subsets of 3 elements of X in such a way that, two disjoint subsets have distinct colors.
Prove that:
(a) 4 colors are sufficient;
(b) 3 colors are not sufficient.