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by Sayan

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1 On the sides of a triangle ABC right angled at A three points D, E and F (respectively BC, AC and AB) are chosen so that the quadrilateral $AFDE$ is a square. If x is the length of the side of the square, show that

$$\frac{1}{x} = \frac{1}{AB} + \frac{1}{AC}$$

2 Determine all positive integers that are equal to 300 times the sum of their digits.

3 Let n be an integer greater than or equal to 2. There are n people in one line, each of which is either a *scoundrel* (who always lie) or a *knight* (who always tells the truth). Every person, except the first, indicates a person in front of him/her and says "This person is a scoundrel" or "This person is a knight." Knowing that there are strictly more scoundrel than knights, seeing the statements show that it is possible to determine each person whether he/she is a scoundrel or a knight.

4 Let $x_1, x_2, x_3, \cdots$ be a sequence defined by the following recurrence relation:

$$\begin{cases} x_1 & = 4 \\ x_{n+1} & = x_1 x_2 x_3 \cdots x_n + 5 \text{ for } n \geq 1 \end{cases}$$

The first few terms of the sequence are $x_1 = 4, x_2 = 9, x_3 = 41 \cdots$

Find all pairs of positive integers $\{a, b\}$ such that $x_a x_b$ is a perfect square.

5 $ABCD$ is a square. Describe the locus of points P, different from A, B, C, D, on that plane for which

$$\widehat{APB} + \widehat{CPD} = 180°$$

6 Determine all pairs $\{a, b\}$ of positive integers with the property that, in whatever manner you color the positive integers with two colors A and B, there always exist two positive integers of color A having their difference equal to a **or** of color B having their difference equal to b.