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1 A model car is tested on some closed circuit 600 meters long, consisting of flat stretches, uphill and downhill. All uphill and downhill have the same slope. The test highlights the following facts:

(a) The velocity of the car depends only on the fact that the car is driving along a stretch of uphill, plane or downhill; calling these three velocities v_s, v_p and v_d respectively, we have $v_s < v_p < v_d$;

(b) v_s, v_p and v_d, expressed in meter per second, are integers.

(c) Whatever may be the structure of the circuit, the time taken to complete the circuit is always 50 seconds.

Find all possible values of v_s, v_p and v_d.

2 In triangle ABC, suppose we have $a > b$, where $a = BC$ and $b = AC$. Let M be the midpoint of AB, and α, β are inscircles of the triangles ACM and BCM respectively. Let then A' and B' be the points of tangency of α and β on CM. Prove that $A'B' = \frac{a-b}{2}$.

3 Each integer is colored with one of two colors, red or blue. It is known that, for every finite set A of consecutive integers, the absolute value of the difference between the number of red and blue integers in the set A is at most 1000. Prove that there exists a set of 2000 consecutive integers in which there are exactly 1000 red numbers and 1000 numbers blue.

4 $\overline{5654}_b$ is a power of a prime number. Find b if $b > 6$.

5 ABC is an isosceles triangle with $AB = AC$ and the angle in A is less than 60°. Let D be a point on AC such that $\angle DBC = \angle BAC$. E is the intersection between the perpendicular bisector of BD and the line parallel to BC passing through A. F is a point on the line AC such that $FA = 2AC$ (A is between F and C).
Show that EB and AC are parallel and that the perpendicular from F to AB, the perpendicular from E to AC and BD are concurrent.

6 Two magicians are performing the following game. Initially the first magician encloses the second magician in a cabin where he can neither see nor hear anything. To start the game, the

first magician invites Daniel, from the audience, to put on each square of a chessboard $n \times n$, at his (Daniel's) discretion, a token black or white. Then the first magician asks Daniel to show him a square C of his own choice. At this point, the first magician chooses a square D (not necessarily different from C) and replaces the token that is on D with other color token (white with black or black with white).
Then he opens the cabin in which the second magician was held. Looking at the chessboard, the second magician guesses what is the square C. For what value of n, the two magicians have a strategy such that the second magician makes a successful guess.