ITAMO 2014

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1	For every 3-digit natural number n (leading digit of n is nonzero), we consider the number n_0 obtained from n eliminating all possible digits that are zero. For example, if $n = 207$, then $n_0 = 27$. Determine the number of three-digit positive integers n, for which n_0 is a divisor of n different from n.
2	Let ABC be a triangle. Let H be the foot of the altitude from C on AB. Suppose that $AH = 3HB$. Suppose in addition we are given that (a) M is the midpoint of AB; (b) N is the midpoint of AC; (c) P is a point on the opposite side of B with respect to the line AC such that $NP = NC$ and $PC = CB$. Prove that $\angle APM = \angle PBA$.
3	For any positive integer n, let D_n denote the greatest common divisor of all numbers of the form $a^n + (a+1)^n + (a+2)^n$ where a varies among all positive integers. (a) Prove that for each n, D_n is of the form 3^k for some integer $k \geq 0$. (b) Prove that, for all $k \geq 0$, there exists an integer n such that $D_n = 3^k$.
4	Let ω be a circle with center A and radius R. On the circumference of ω four distinct points B, C, G, H are taken in that order in such a way that G lies on the extended B-median of the triangle ABC, and H lies on the extension of altitude of ABC from B. Let X be the intersection of the straight lines AC and GH. Show that the segment AX has length $2R$.
5	Prove that there exists a positive integer that can be written, in at least two ways, as a sum of 2014-th powers of 2015 distinct positive integers $x_1 < x_2 < \cdots < x_{2015}$.
6	A $(2n+1) \times (2n+1)$ grid, with $n > 0$, is colored in such a way that each of the cell is white or black. A cell is called *special* if there are at least n other cells of the same color in its row, and at least another n cells of the same color in its column. (a) Prove that there are at least $2n+1$ special boxes. (b) Provide an example where there are at most $4n$ special cells. (c) Determine, as a function of n, the minimum possible number of special cells.