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by lssl

1 In a concyclic quadrilateral $PQRS,\angle PSR=\frac{\pi}{2}$, H,K are perpendicular foot from Q to sides PR,RS , prove that HK bisect segmentSQ.

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2 The underside of a pyramid is a convex nonagon , paint all the diagonals of the nonagon and all the ridges of the pyramid into white and black , prove : there exists a triangle ,the colour of its three sides are the same . (PS:the sides of the nonagon is not painted.)

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3 Given s,t are non-zero integers, (x,y) is an integer pair , A transformation is to change pair (x,y) into pair $(x+t,y-s)$. If the two integers in a certain pair becoems relatively prime after several tranfomations , then we call the original integer pair "a good pair" .

(1) Is (s,t) a good pair ?

(2) Prove :for any s and t , there exists pair (x,y) which is " a good pair".

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4 Define a function f on positive real numbers to satisfy

$$f(1)=1,f(x+1)=xf(x)\text{ and }f(x)=10^{g(x)},$$

where $g(x)$ is a function defined on real numbers and for all real numbers y,z and $0\le t\le 1$, it satisfies

$$g(ty+(1-t)z)\le tg(y)+(1-t)g(z).$$

(1) Prove: for any integer n and $0\le t\le 1$, we have

$$t[g(n)-g(n-1)]\le g(n+t)-g(n)\le t[g(n+1)-g(n)].$$

(2) Prove that

$$\frac{4}{3}\le f(\frac{1}{2})\le\frac{4}{3}\sqrt{2}.$$
