Hong kong National Olympiad 2004

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1 Let $a_1, a_2, ..., a_{n+1} (n > 1)$ are positive real numbers such that $a_2 - a_1 = a_3 - a_2 = ... = a_{n+1} - a_n$. Prove that $\sum_{k=2}^{n} \frac{1}{a_k^2} \leq \frac{n-1}{2} . \frac{a_1 a_n + a_2 a_{n+1}}{a_1 a_2 a_n a_{n+1}}$

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2 In a school there b teachers and c students. Suppose that
a) each teacher teaches exactly k students, and
b)for any two (distinct) students , exactly h teachers teach both of them.
Prove that $\frac{b}{h} = \frac{c(c-1)}{k(k-1)}$.

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3 Points P and Q are taken sides AB and AC of a triangle ABC respectively such that $A\hat{P}C = A\hat{Q}B = 45^0$. The line through P perpendicular to AB intersects BQ at S, and the line through Q perpendicular to AC intersects CP at R. Let D be the foot of the altitude of triangle ABC from A. Prove that $SR \parallel BC$ and PS, AD, QR are concurrent.

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4 Let $S = \{1, 2, ..., 100\}$. Find number of functions $f : S \to S$ satisfying the following conditions
a)$f(1) = 1$
b)f is bijective
c)$f(n) = f(g(n))f(h(n)) \forall n \in S$, where $g(n), h(n)$ are positive integer numbers such that $g(n) \leq h(n), n = g(n)h(n)$ that minimize $h(n) - g(n)$.
