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1 A subset M of $\{1, 2, ..., 2006\}$ has the property that for any three elements x, y, z of M with $x < y < z$, $x + y$ does not divide z. Determine the largest possible size of M.

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2 For a positive integer k, let $f_1(k)$ be the square of the sum of the digits of k. Define f_{n+1} = $f_1 \circ f_n$. Evaluate $f_{2007}(2^{2006})$.

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3 A convex quadrilateral $ABCD$ with $AC \neq BD$ is inscribed in a circle with center O. Let E be the intersection of diagonals AC and BD. If P is a point inside $ABCD$ such that $\angle PAB + \angle PCB = \angle PBC + \angle PDC = 90^\circ$, prove that O, P and E are collinear.

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4 Let $(a_n)_{n\geq 1}$ be a sequence of positive numbers. If there is a constant $M > 0$ such that $a_2^2 + a_2^2 + \ldots + a_n^2 < M a_{n+1}^2$ for all n, then prove that there is a constant $M' > 0$ such that $a_1 + a_2 + \ldots + a_n < M' a_{n+1}$.
