Hong kong National Olympiad 2008

www.artofproblemsolving.com/community/c5415

by brianchung11

1 Let $f(x) = c_mx^m + c_{m-1}x^{m-1} + ... + c_1x + c_0$, where each c_i is a non-zero integer. Define a sequence $\{a_n\}$ by $a_1 = 0$ and $a_{n+1} = f(a_n)$ for all positive integers n.

(a) Let i and j be positive integers with $i < j$. Show that $a_{j+1} - a_j$ is a multiple of $a_{i+1} - a_i$.

(b) Show that $a_{2008} \neq 0$

---

2 Let $n > 4$ be a positive integer such that n is composite (not a prime) and divides $\varphi(n)\sigma(n)+1$, where $\varphi(n)$ is the Euler's totient function of n and $\sigma(n)$ is the sum of the positive divisors of n. Prove that n has at least three distinct prime factors.

---

3 ΔABC is a triangle such that $AB \neq AC$. The incircle of ΔABC touches BC, CA, AB at D, E, F respectively. H is a point on the segment EF such that $DH \bot EF$. Suppose $AH \bot BC$, prove that H is the orthocentre of ΔABC.

Remark: the original question has missed the condition $AB \neq AC$

---

4 There are 2008 congruent circles on a plane such that no two are tangent to each other and each circle intersects at least three other circles. Let N be the total number of intersection points of these circles. Determine the smallest possible values of N.
