Saint Petersburg Mathematical Olympiad 2012

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–	**Grade 11**
1	a,b,c are reals, such that every pair of equations of $x^3-ax^2+b=0, x^3-bx^2+c=0, x^3-cx^2+a=0$ has common root. Prove $a=b=c$
2	We have big multivolume encyclopaedia about dogs on the shelf in alphabetical order, each volume in its specially selected place. Near each place there is an instruction that prescribes one of four actions: to rearrange this volume is one or two places left or right. If you simultaneously run all instructions, volumes will be placed in the same places in another order. The cynologist Dima performs all the instructions every morning. Once he discovered, that the volume of ”Bichons” stands still, which was initially occupied by the volume of ”Terriers”. Prove , that after some time the volume of ”Mudies” will stand on the original place of the volume ”Poodles”.
3	At the base of the pyramid $SABCD$ lies a convex quadrilateral $ABCD$, such that $BC*AD=BD*AC$. Also $\angle ADS=\angle BDS, \angle ACS=\angle BCS$. Prove that the plane SAB is perpendicular to the plane of the base.
4	$x_1,...,x_n$ are reals and $x_1^2+...+x_n^2=1$ Prove, that exists such $y_1,...,y_n$ and $z_1,...,z_n$ such that $\|y_1\|+...+\|y_n\| \leq 1; max(\|z_1\|,...,\|z_n\|) \leq 1$ and $2x_i=y_i+z_i$ for every i
5	$n \geq k$ -two natural numbers. S -such natural, that have not less than n divisors. All divisors of S are written in descending order. What minimal number of divisors can have number from k-th place ?
6	On the coordinate plane in the first quarter there are 100 non-intersecting single unit segments parallel to the coordinate axes. These segments aremirrors (on both sides), they reflect the light according to the rule. ”The angle of incidence is equal to the angle of reflection.” (If you hit the edge of the mirror, the beam of light does not change its direction.) From the point lying in the unit circle with the center at the origin, a ray of light in the direction of the bisector of the first coordinate angle. Prove that, that this initial point can be chosen so that the ray is reflected from the mirrors not more than 150 times.

7 Some cities of Russia are connected with some cities of Ukraine with international airlines. The Interstate Council for the Promotion of Migration intends to introduce a one-way traffic on each airline so that, by taking off from the city, it could no longer be returned in this city (using other one-way airlines). Prove that the number of ways to do this is not divided by 3.

– **Grade 10**

1 $\begin{cases} x^3-ax^2+b^3=0 \\ x^3-bx^2+c^3=0 \\ x^3-cx^2+a^3=0 \end{cases}$
Prove that system hasn't solutions if a,b,c are different.

2 Points C,D are on side BE of triangle ABE, such that $BC=CD=DE$. Points X,Y,Z,T are circumcenters of ABE,ABC,ADE,ACD. Prove, that T - centroid of XYZ

3 25 students are on exams. Exam consists of some questions with 5 variants of answer. Every two students gave same answer for not more than 1 question. Prove, that there are not more than 6 questions in exam.

4 Some notzero reals numbers are placed around circle. For every two neighbour numbers a,b it true, that $a+b$ and $\frac{1}{a}+\frac{1}{b}$ are integer. Prove that there are not more than 4 different numbers.

5 S is natural, and $S=d_1>d_2>...>d_{1000000}=1$ are all divisors of S. What minimal number of divisors can have d_{250}?

6 $ABCD$ is parallelogram. Line l is perpendicular to BC at B. Two circles passes through D,C, such that l is tangent in points P and Q. M - midpoint AB.
Prove that $\angle DMP=\angle DMQ$

7 Same as Grade 11 P6

– **Grade 9**

1 Find all integer b such that $[x^2]-2012x+b=0$ has odd number of roots.

2 Natural a,b,c are >100 and $(a,b,c)=1$. $c|a+b, a|b+c$ Find minimal b

3 $ABCD$ is inscribed. Bisector of angle between diagonals intersect AB anc CD at X and Y. M,N are midpoints of AD,BC. $XM=YM$ Prove, that $XN=YN$.

4 Same as Grade10 P3

5 In the 100×100 table in every cell there is natural number. All numbers in same row or column are different.
Can be that for every square sum of numbers, that are in angle cells, is square number ?

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6 ABC is triangle. Point L is inside ABC and lies on bisector of $\angle B$. K is on BL. $\angle KAB = \angle LCB = \alpha$. Point P inside triangle is such, that $AP = PC$ and $\angle APC = 2\angle AKL$.
Prove that $\angle KPL = 2\alpha$

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7 We have 2012 sticks with integer length, and sum of length is n. We need to have sticks with lengths $1, 2, ..., 2012$. For it we can break some sticks (for example from stick with length 6 we can get 1 and 4).
For what minimal n it is always possible?
