Saint Petersburg Mathematical Olympiad 2013

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by RagvaloD, mathuz

–   **Grade 11**

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1   Find the minimum positive noninteger root of $\sin x = \sin\lfloor x \rfloor$.

F. Petrov

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2   At the faculty of mathematics study 40 boys and 10 girls. Every girl acquaintance with all boys, who older than her, or tall(higher) than her. Prove that there exist two boys such that the sets of acquainted-girls of the boys are same.

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3   Let M and N are midpoint of edges AB and CD of the tetrahedron $ABCD$, $AN = DM$ and $CM = BN$. Prove that $AC = BD$.

S. Berlov

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4   Find all pairs (p, q) of prime numbers, such that $2p - 1, 2q - 1, 2pq - 1$ are perfect square.

F. Petrov, A. Smirnov

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5   Let $x_1, \ldots, x_{n+1} \in [0, 1]$ and $x_1 = x_{n+1}$. Prove that

$$\prod_{i=1}^{n}(1 - x_i x_{i+1} + x_i^2) \geq 1.$$

A. Khrabrov, F. Petrov

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6   Let $(I_b), (I_c)$ are excircles of a triangle ABC. Given a circle ω passes through A and externally tangents to the circles (I_b) and (I_c) such that it intersects with BC at points M, N. Prove that $\angle BAM = \angle CAN$.

A. Smirnov

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7   In the language of wolves has two letters F and P, any finite sequence which forms a word. word Y is called 'subpart' of word X if Y is obtained from X by deleting some letters (for example, the word $FFPF$ has 8 'subpart's: F, P, FF, FP, PF, FFP, FPF, FFF). Determine n such that the n is the greatest number of 'subpart's can have n-letter word language of wolves.

F. Petrov, V. Volkov

– Grade 10

1 Call number A as interesting if A is divided by every number that can be received from A by crossing some last digits. Find maximum interesting number with different digits.

2 in a convex quadrilateral $ABCD$, M, N are midpoints of BC, AD respectively. If $AM = BN$ and $DM = CN$ then prove that $AC = BD$.

S. Berlov

3 On a circle there are some black and white points (there are at least 12 points). Each point has 10 neighbors (5 left and 5 right neighboring points), 5 being black and 5 white. Prove that the number of points on the circle is divisible by 4.

4 There are 100 numbers from $(0, 1)$ on the board. On every move we replace two numbers a, b with roots of $x^2 - ax + b = 0$(if it has two roots). Prove that process is not endless.

5 Given quadrilateral $ABCD$ with $AB = BC = CD$. Let $AC \cap BD = O, X, Y$ are symmetry points of O respect to midpoints of BC, AD, and Z is intersection point of lines, which perpendicular bisects of AC, BD. Prove that X, Y, Z are collinear.

6 There are 85 soldiers with different heigth and age. Every day commander chooses random soldier and send him and also all soldiers that are taller and older than this soldier, or all soldiers that are lower and younger than this soldier to color grass.
Prove that after 10 days we can find two soldiers, that color grass at same days.

7 Let a_1, a_2 - two naturals, and $1 < b_1 < a_1, 1 < b_2 < a_2$ and $b_1|a_1, b_2|a_2$. Prove that $a_1b_1 + a_2b_2 - 1$ is not divided by a_1a_2

– Grade 9

1 Same as Grade 10 P1

2 if $a^2 + b^2 + c^2 + d^2 = 1$ prove that

$$(1-a)(1-b) \geq cd.$$

A. Khrabrov

3 ABC is triangle. l_1- line passes through A and parallel to BC, l_2 - line passes through C and parallel to AB. Bisector of $\angle B$ intersect l_1 and l_2 at X, Y. $XY = AC$. What value can take $\angle A - \angle C$?

4   There are 100 glasses, with $101, 102, ..., 200$ cents. Two players play next game. In every move they can take some cents from one glass, but after move should be different number of cents in every glass. Who will win with right strategy?

5   Same as Grade10 P4

6   Same as Grade10 P5

7   Given is a natural number a with 54 digits, each digit equal to 0 or 1. Prove the remainder of a when divide by $33 \cdot 34 \cdots 39$ is larger than 100000.
(It's mean: $a \equiv r \pmod{33 \cdot 34 \cdots 39}$ with $0 < r < 33 \cdot 34 \cdots 39$ then prove that $r > 100000$)
M. Antipov