Hong kong National Olympiad 2013

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by WakeUp

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1	Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that $$\sqrt[4]{\frac{\sqrt{3}}{a} + 6\sqrt{3}b} + \sqrt[4]{\frac{\sqrt{3}}{b} + 6\sqrt{3}c} + \sqrt[4]{\frac{\sqrt{3}}{c} + 6\sqrt{3}a} \le \frac{1}{abc}$$ When does inequality hold?
2	For any positive integer a, define $M(a)$ to be the number of positive integers b for which $a+b$ divides ab. Find all integer(s) a with $1 \le a \le 2013$ such that $M(a)$ attains the largest possible value in the range of a.
3	Let ABC be a triangle with $CA > BC > AB$. Let O and H be the circumcentre and orthocentre of triangle ABC respectively. Denote by D and E the midpoints of the arcs AB and AC of the circumcircle of triangle ABC not containing the opposite vertices. Let D' be the reflection of D about AB and E' the reflection of E about AC. Prove that O, H, D', E' are concylic if and only if A, D', E' are collinear.
4	In a chess tournament there are $n > 2$ players. Every two players play against each other exactly once. It is known that exactly n games end as a tie. For any set S of players, including A and B, we say that A *admires* B *in that set* if i) A does not beat B; or ii) there exists a sequence of players $C_1, C_2, \ldots, C_k$ in S, such that A does not beat C_1, C_k does not beat B, and C_i does not beat C_{i+1} for $1 \le i \le k-1$. A set of four players is said to be *harmonic* if each of the four players admires everyone else in the set. Find, in terms of n, the largest possible number of harmonic sets.