Hong kong Team Selection Test 1994

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Day 1

1 In a $\triangle ABC$, $\angle C = 2\angle B$. P is a point in the interior of $\triangle ABC$ satisfying that $AP = AC$ and $PB = PC$. Show that AP trisects the angle $\angle A$.

2 In a table-tennis tournament of 10 contestants, any 2 contestants meet only once.
We say that there is a winning triangle if the following situation occurs: i-th contestant defeated the j-th contestant, j-th contestant defeated the k-th contestant, and, k-th contestant defeated the i-th contestant.
Let, W_i and L_i be respectively the number of games won and lost by the i-th contestant.
Suppose, $L_i + W_j \geq 8$ whenever the j-th contestant defeats the i-th contestant.
Prove that, there are exactly 40 winning triangles in this tournament.

3 Find all non-negative integers x, y and z satisfying the equation:

$$7^x + 1 = 3^y + 5^z$$

Day 2

1 Suppose, $x, y, z \in \mathbb{R}_+$ such that $xy + yz + zx = 1$. Prove that,

$$x(1-y^2)(1-z^2) + y(1-z^2)(1-x^2) + z(1-x^2)(1-y^2) \leq \frac{4\sqrt{3}}{9}$$

2 Given that, a function $f(n)$, defined on the natural numbers, satisfies the following conditions:
(i)$f(n) = n - 12$ if $n > 2000$; (ii)$f(n) = f(f(n+16))$ if $n \leq 2000$.
(a) Find $f(n)$.
(b) Find all solutions to $f(n) = n$.

3 Let m and n be positive integers where m has d digits in base ten and $d \leq n$. Find the sum of all the digits (in base ten) of the product $(10^n - 1)m$.