Hong kong Team Selection Test 2006

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1   On a planet there are $3 \times 2005!$ aliens and 2005 languages. Each pair of aliens communicates with each other in exactly one language. Show that there are 3 aliens who communicate with each other in one common language.

2   Suppose there are $4n$ line segments of unit length inside a circle of radius n. Furthermore, a straight line L is given. Prove that there exists a straight line L' that is either parallel or perpendicular to L and that L' cuts at least two of the given line segments.

3   Let a, b, c, d be positive real numbers such that $a + b + c + d = 1$. Prove that

$$6(a^3 + b^3 + c^3 + d^3) \geq (a^2 + b^2 + c^2 + d^2) + \frac{1}{8}$$

4   Show that there exist infinitely many square-free positive integers n that divide $2005^n - 1$.