Hong kong Team Selection Test 2008

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– Test 1

Day 1

1 Let α_1, α_2, ..., α_{2008} be real numbers. Find the maximum value of

$$\sin\alpha_1\cos\alpha_2 + \sin\alpha_2\cos\alpha_3 + \cdots + \sin\alpha_{2007}\cos\alpha_{2008} + \sin\alpha_{2008}\cos\alpha_1$$

2 Find the total number of solutions to the following system of equations:

$$\begin{cases} a^2 + bc \equiv a \pmod{37} \\ b(a+d) \equiv b \pmod{37} \\ c(a+d) \equiv c \pmod{37} \\ bc + d^2 \equiv d \pmod{37} \\ ad - bc \equiv 1 \pmod{37} \end{cases}$$

3 Let $ABCDE$ be an arbitrary convex pentagon. Suppose that $BD \cap CE = A'$, $CE \cap DA = B'$, $DA \cap EB = C'$, $EB \cap AC = D'$ and $AC \cap BD = E'$. Suppose also that $(ABD') \cap (AC'E) = A''$, $(BCE') \cap (BD'A) = B''$, $(CDA') \cap (CE'B) = C''$, $(DEB') \cap DA'C = D''$ and $(EAC') \cap (EB'D) = E''$. Prove that AA'', BB'', CC'', DD'' and EE'' are concurrent.

Day 2

1 In a school there are 2008 students. Students are members of certain committees. A committee has at most 1004 members and every two students join a common committee.
(i) Determine the smallest possible number of committees in the school.
(ii) If it is further required that the union of any two committees consists of at most 1800 students, will your answer in (i) still hold?

2 Let a, b, c be the three sides of a triangle. Determine all possible values of

$$\frac{a^2 + b^2 + c^2}{ab + bc + ca}$$

3 Show that the equation $y^{37} = x^3 + 11 \pmod p$ is solvable for every prime p, where $p \leq 100$.

– Grade level 2

1 Let $f : Z \to Z$ be such that $f(1) = 1, f(2) = 20, f(-4) = -4$ and $f(x+y) = f(x) + f(y) + axy(x+y) + bxy + c(x+y) + 4 \forall x, y \in Z$, where a, b, c are constants.

(a) Find a formula for $f(x)$, where x is any integer.
(b) If $f(x) \geq mx^2 + (5m+1)x + 4m$ for all non-negative integers x, find the greatest possible value of m.

2 Define a k-*clique* to be a set of k people such that every pair of them are acquainted with each other. At a certain party, every pair of 3-cliques has at least one person in common, and there are no 5-cliques. Prove that there are two or fewer people at the party whose departure leaves no 3-clique remaining.

3 Prove that there are infinitely many primes p such that the total number of solutions mod p to the equation $3x^3 + 4y^4 + 5z^3 - y^4 z \equiv 0$ is p^2

4 Two circles C_1, C_2 with different radii are given in the plane, they touch each other externally at T. Consider any points $A \in C_1$ and $B \in C_2$, both different from T, such that $\angle ATB = 90^\circ$.

(a) Show that all such lines AB are concurrent.
(b) Find the locus of midpoints of all such segments AB.