National Olympiad Second Round 1993

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1 Prove that there is a number such that its decimal represantation ends with 1994 and it can be written as $1994 \cdot 1993^n$ ($n \in Z^+$)

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2 I centered incircle of triangle ABC $(m(\hat{B}) = 90^\circ)$ touches $[AB], [BC], [AC]$ respectively at F, D, E. $[CI] \cap [EF] = L$ and $[DL] \cap [AB] = N$. Prove that $[AI] = [ND]$.

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3 $n \in Z^+$ and $A = 1, \dots, n$. $f : N \to N$ and $\sigma : N \to N$ are two permutations, if there is one $k \in A$ such that $(f \circ \sigma)(1), \dots, (f \circ \sigma)(k)$ is increasing and $(f \circ \sigma)(k), \dots, (f \circ \sigma)(n)$ is decreasing sequences we say that f is good for σ. S_σ shows the set of good functions for σ.
a) Prove that, S_σ has got 2^{n-1} elements for every σ permutation.
b)$n \geq 4$, prove that there are permutations σ and τ such that, $S_\sigma \cap S_\tau = \phi$
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4 a_n is a sequence of positive integers such that, for every $n \geq 1$, $0 < a_{n+1} - a_n < \sqrt{a_n}$. Prove that for every $x, y \in R$ such that $0 < x < y < 1$ $x < \frac{a_k}{a_m} < y$ we can find such $k, m \in Z^+$.

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5 Prove that we can draw a line (by a ruler and a compass) from a vertice of a convex quadrilateral such that, the line divides the quadrilateral to two equal areas.

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6 $n_1, \dots, n_k, a$ are integers that satisfies the above conditions A)For every $i \neq j$, $(n_i, n_j) = 1$ B)For every $i, a^{n_i} \equiv 1 (mod n_i)$ C)For every $i, X^{a-1} \equiv 0 (mod n_i)$.
Prove that $a^x \equiv 1 (mod x)$ congruence has at least $2^{k+1} - 2$ solutions. ($x > 1$)
