National Olympiad Second Round 1994

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by xeroxia

1 For $n \in \mathbb{N}$, let a_n denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{a_n^3}.$$

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2 Let $ABCD$ be a cyclic quadrilateral $\angle BAD < 90^\circ$ and $\angle BCA = \angle DCA$. Point E is taken on segment DA such that $BD = 2DE$. The line through E parallel to CD intersects the diagonal AC at F. Prove that

$$\frac{AC \cdot BD}{AB \cdot FC} = 2.$$

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3 Let n blue lines, no two of which are parallel and no three concurrent, be drawn on a plane. An intersection of two blue lines is called a blue point. Through any two blue points that have not already been joined by a blue line, a red line is drawn. An intersection of two red lines is called a red point, and an intersection of red line and a blue line is called a purple point. What is the maximum possible number of purple points?

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4 Let $f : \mathbb{R}^+ \to \mathbb{R}+$ be an increasing function. For each $u \in \mathbb{R}^+$, we denote $g(u) = \inf\{f(t) + u/t \mid t > 0\}$. Prove that:

(a) If $x \leq g(xy)$, then $x \leq 2f(2y)$;

(b) If $x \leq f(y)$, then $x \leq 2g(xy)$.

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5 Find the set of all ordered pairs (s, t) of positive integers such that

$$t^2 + 1 = s(s + 1).$$

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6 The incircle of triangle ABC touches BC at D and AC at E. Let K be the point on CB with $CK = BD$, and L be the point on CA with $AE = CL$. Lines AK and BL meet at P. If Q is the midpoint of BC, I the incenter, and G the centroid of $\triangle ABC$, show that:

(a) IQ and AK are parallel,

(b) the triangles AIG and QPG have equal area.
