**National Olympiad Second Round 1996**

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**Day 1** December 6th

**1** Let $(A_n)_{n=1}^{\infty}$ and $(a_n)_{n=1}^{\infty}$ be sequences of positive integers. Assume that for each positive integer $x$, there is a unique positive integer $N$ and a unique $N-tuple$ $(x_1,...,x_N)$ such that $0\leq x_k\leq a_k$ for $k=1,2,...N$, $x_N\neq 0$, and $x=\sum\limits_{k=1}^{N}A_kx_k$.

(a) Prove that $A_k=1$ for some $k$;
(b) Prove that $A_k=A_j\Leftrightarrow k=j$;
(c) Prove that if $A_k\leq A_j$, then $A_k|\,A_j$.

**2** Let $ABCD$ be a square of side length 2, and let $M$ and $N$ be points on the sides $AB$ and $CD$ respectively. The lines $CM$ and $BN$ meet at $P$, while the lines $AN$ and $DM$ meet at $Q$. Prove that $|PQ|\geq 1$.

**3** Let $n$ integers on the real axis be colored. Determine for which positive integers $k$ there exists a family $K$ of closed intervals with the following properties:
i) The union of the intervals in $K$ contains all of the colored points;
ii) Any two distinct intervals in $K$ are disjoint;
iii) For each interval $I$ at $K$ we have $a_I=k.b_I$, where $a_I$ denotes the number of integers in $I$, and $b_I$ the number of colored integers in $I$.

**Day 2** December 7th

**1** A circle is tangent to sides $AD,\ DC,\ CB$ of a convex quadrilateral $ABCD$ at K, L, M respectively. A line $l$, passing through $L$ and parallel to $AD$, meets $KM$ at $N$ and $KC$ at $P$. Prove that $PL=PN$.

**2** Prove that $\prod\limits_{k=0}^{n-1}(2^n-2^k)$ is divisible by $n!$ for all positive integers $n$.

**3** Show that there is no function $f:\mathbb{R}^+\to\mathbb{R}^+$ such that $f(x+y)>f(x)(1+yf(x))$ for all $x,y\in\mathbb{R}^+$.