National Olympiad Second Round 1999

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Day 1 December 3rd

1 Find the number of ordered quadruples (x, y, z, w) of integers with $0 \leq x, y, z, w \leq 36$ such that $x^2 + y^2 \equiv z^3 + w^3$ (mod 37).

2 Problem-2:
Given a circle with center O, the two tangent lines from a point S outside the circle touch the circle at points P and Q. Line SO intersects the circle at A and B, with B closer to S. Let X be an interior point of minor arc PB, and let line OS intersect lines QX and PX at C and D, respectively. Prove that
$\frac{1}{|AC|} + \frac{1}{|AD|} = \frac{2}{|AB|}$.

3 For any two positive integers n and p, prove that there are exactly $(p+1)^{n+1} - p^{n+1}$ functions
$f : \{1, 2, ..., n\} \to \{-p, -p+1, -p+2, ..., p-1, p\}$
such that $|f(i) - f(j)| \leq p$ for all $i, j \in \{1, 2, ..., n\}$.

Day 2 December 4th

4 Find all sequences $a_1, a_2, ..., a_{2000}$ of real numbers such that $\sum\limits_{n=1}^{2000} a_n = 1999$ and such that $\frac{1}{2} < a_n < 1$ and $a_{n+1} = a_n(2 - a_n)$ for all $n \geq 1$.

5 In an acute triangle $\triangle ABC$ with circumradius R, altitudes $\overline{AD}, \overline{BE}, \overline{CF}$ have lengths h_1, h_2, h_3, respectively. If t_1, t_2, t_3 are lengths of the tangents from A, B, C, respectively, to the circumcircle of triangle $\triangle DEF$, prove that
$\sum\limits_{i=1}^{3} \left(\frac{t_i}{\sqrt{h_i}}\right)^2 \leq \frac{3}{2}R.$

6 We wish to nd the sum of 40 given numbers utilizing 40 processors. Initially, we have the number 0 on the screen of each processor. Each processor adds the number on its screen with a number entered directly (only the given numbers could be entered directly to the processors) or transferred from another processor in a unit time. Whenever a number is transferred from a processor to another, the former processor resets. Find the least time needed to nd the desired sum.