JBMO Shortlist 2016

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– Algebra

1 Let a, b, c be positive real numbers such that $abc = 8$. Prove that $\frac{ab+4}{a+2} + \frac{bc+4}{b+2} + \frac{ca+4}{c+2} \geq 6$.

2 Let a, b, cbe positive real numbers.Prove that

$\frac{8}{(a+b)^2+4abc} + \frac{8}{(b+c)^2+4abc} + \frac{8}{(a+c)^2+4abc} + a^2 + b^2 + c^2 \geq \frac{8}{a+3} + \frac{8}{b+3} + \frac{8}{c+3}$.

3 Find all the pairs of integers (m, n) such that $\sqrt{n + \sqrt{2016}} + \sqrt{m - \sqrt{2016}} \in \mathbb{Q}$.

4 If the non-negative reals x, y, z satisfy $x^2 + y^2 + z^2 = x + y + z$. Prove that

$$\frac{x+1}{\sqrt{x^5+x+1}} + \frac{y+1}{\sqrt{y^5+y+1}} + \frac{z+1}{\sqrt{z^5+z+1}} \geq 3.$$

When does the equality occur?

Proposed by Dorlir Ahmeti, Albania

5 Let x, y, z be positive real numbers such that $x + y + z = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$. Prove that

$$x + y + z \geq \sqrt{\frac{xy+1}{2}} + \sqrt{\frac{yz+1}{2}} + \sqrt{\frac{zx+1}{2}}\ .$$

Proposed by Azerbaijan

Let x, y, z be positive real numbers such that $x + y + z = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$. Prove that

$$x + y + z \geq \sqrt{\frac{x^2+1}{2}} + \sqrt{\frac{y^2+1}{2}} + \sqrt{\frac{z^2+1}{2}}\ .$$

– Combinatorics

1 Let S_n be the sum of reciprocal values of non-zero digits of all positive integers up to (and including) n. For instance, $S_{13} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{3}$. Find the least positive integer k making the number $k! \cdot S_{2016}$ an integer.

2 The natural numbers from 1 to 50 are written down on the blackboard. At least how many of them should be deleted, in order that the sum of any two of the remaining numbers is not a prime?

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3 A 5×5 table is called regular f each of its cells contains one of four pairwise distinct real numbers,such that each of them occurs exactly one in every 2×2 subtable.The sum of all numbers of a regular table is called the total sum of the table.With any four numbers,one constructs all possible regular tables,computes their total sums and counts the distinct outcomes.Determine the maximum possible count.

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4 A splitting of a planar polygon is a fi nite set of triangles whose interiors are pairwise disjoint, and whose union is the polygon in question. Given an integer $n \geq 3$, determine the largest integer m such that no planar n-gon splits into less than m triangles.

– Geometry

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1 Let ABC be an acute angled triangle, let O be its circumcentre, and let D, E, F be points on the sides BC, CA, AB, respectively. The circle (c_1) of radius FA, centered at F, crosses the segment OA at A' and the circumcircle (c) of the triangle ABCagain at K. Similarly, the circle (c_2) of radius DB, centered at D, crosses the segment (OB) at B' and the circle (c) again at L. Finally, the circle (c_3) of radius EC, centered at E, crosses the segment (OC)at C' and the circle (c) again at M. Prove that the quadrilaterals $BKFA'$, $CLDB'$ and $AMEC'$ are all cyclic, and their circumcircles share a common point.

Evangelos Psychas (Greece)

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2 Let ABC be a triangle with $\angle BAC = 60^{\circ}$. Let D and E be the feet of the perpendiculars from A to the external angle bisectors of $\angle ABC$ and $\angle ACB$, respectively. Let O be the circumcenter of the triangle ABC. Prove that the circumcircles of the triangles $ADE$and BOC are tangent to each other.

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3 A trapezoid $ABCD$ ($AB||CF$,$AB > CD$) is circumscribed.The incircle of the triangle ABC touches the lines AB and AC at the points M and N,respectively.Prove that the incenter of the trapezoid $ABCD$ lies on the line MN.

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4 Let ABC be an acute angled triangle whose shortest side is BC. Consider a variable point P on the side BC, and let D and E be points on AB and AC, respectively, such that $BD = BP$ and $CP = CE$. Prove that, as P traces BC, the circumcircle of the triangle ADE passes through a fixed point.

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5 Let ABC be an acute angled triangle with orthocenter H and circumcenter O. Assume the circumcenter X of BHC lies on the circumcircle of ABC. Reflect O across X to obtain O', and let the lines $XH$and $O'A$ meet at K. Let L, M and N be the midpoints of $[XB]$, $[XC]$ and

$[BC]$, respectively. Prove that the points K, L, M and N are concyclic.

6 Given an acute triangle ABC, erect triangles ABD and ACE externally, so that $\angle ADB = \angle AEC = 90^o$ and $\angle BAD = \angle CAE$. Let $A_1 \in BC, B_1 \in AC$ and $C_1 \in AB$ be the feet of the altitudes of the triangle ABC, and let K and K, L be the midpoints of $[BC_1]$ and BC_1, CB_1, respectively. Prove that the circumcenters of the triangles $AKL, A_1B_1C_1$ and DEA_1 are collinear.

(Bulgaria)

7 Let AB be a chord of a circle (c) centered at O, and let K be a point on the segment AB such that $AK < BK$. Two circles through K, internally tangent to (c) at A and B, respectively, meet again at L. Let P be one of the points of intersection of the line KL and the circle (c), and let the lines AB and LO meet at M. Prove that the line MP is tangent to the circle (c).

Theoklitos Paragyiou (Cyprus)

– Number Theory

1 Determine the largest positive integer n that divides $p^6 - 1$ for all primes $p > 7$.

2 Find the maximum number of natural numbers $x_1, x_2, ..., x_m$ satisfying the conditions:
a) No $x_i - x_j, 1 \le i < j \le m$ is divisible by 11, and
b) The sum $x_2x_3...x_m + x_1x_3...x_m + \cdots + x_1x_2...x_{m-1}$ is divisible by 11.

3 Find all positive integers n such that the number $A_n = \frac{2^{4n+2}+1}{65}$ is
a) an integer,
b) a prime.

4 Find all triplets of integers (a, b, c) such that the number

$$N = \frac{(a-b)(b-c)(c-a)}{2} + 2$$

is a power of 2016.

(A power of 2016 is an integer of form 2016^n,where n is a non-negative integer.)

5 Determine all four-digit numbers $\overline{abcd}$ such that $(a+b)(a+c)(a+d)(b+c)(b+d)(c+d) = \overline{abcd}$: