National Olympiad Second Round 2000

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Day 1 December 8th

1 A circle with center O and a point A in this circle are given. Let P_B is the intersection point of $[AB]$ and the internal bisector of $\angle AOB$ where B is a point on the circle such that B doesn't lie on the line OA, Find the locus of P_B as B varies.

2 Let define $P_n(x) = x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x + 1$ for every positive integer n. Prove that for every positive integer a one can find a positive integer n and polynomials $R(x)$ and $Q(x)$ with integer coefficients such that

$$P_n(x) = [1 + ax + x^2R(x)]Q(x).$$

3 Let $f(x, y)$ and $g(x, y)$ be defined for every $x, y \in \{1, 2, \ldots, 2000\}$ and take different values for at most n ordered pairs of (x, y). For every function pairs $f(x, y), g(x, y)$, when $x \notin X$ and $y \notin Y$, it is always possible to find $1000-$element sets $X, Y \subset \{1, 2, \ldots, 2000\}$ such that $f(x, y) = g(x, y)$. Determine the largest integer that n can take.

Day 2 December 9th

1 Let p be a prime number. $T(x)$ is a polynomial with integer coefficients and degree from the set $\{0, 1, ..., p-1\}$ and such that $T(n) \equiv T(m) (modp)$ for some integers m and n implies that $m \equiv n (modp)$. Determine the maximum possible value of degree of $T(x)$

2 A positive real number a and two rays wich intersect at point A are given. Show that all the circles which pass through A and intersect these rays at points B and C where $|AB|+|AC| = a$ have a common point other than A.

3 Find all continuous functions $f : [0, 1] \to [0, 1]$ for which there exists a positive integer n such that $f^n(x) = x$ for $x \in [0, 1]$ where $f^0(x) = x$ and $f^{k+1} = f(f^k(x))$ for every positive integer k.