National Olympiad Second Round 2001

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Day 1 December 22nd

1 Let $ABCD$ be a convex quadrilateral. The perpendicular bisectors of the sides $[AD]$ and $[BC]$ intersect at a point P inside the quadrilateral and the perpendicular bisectors of the sides $[AB]$ and $[CD]$ also intersect at a point Q inside the quadrilateral. Show that, if $\angle APD = \angle BPC$ then $\angle AQB = \angle CQD$

2 $(x_n)_{-\infty<n<\infty}$ is a sequence of real numbers which satisfies $x_{n+1} = \frac{x_n^2+10}{7}$ for every $n \in \mathbb{Z}$. If there exist a real upperbound for this sequence, find all the values x_0 can take.

3 One wants to distribute n same sized cakes between k people equally by cutting every cake at most once. If the number of positive divisors of n is denoted as $d(n)$, show that the number of different values of k which makes such distribution possible is $n + d(n)$

Day 2 December 23rd

1 Find all ordered triples of positive integers (x, y, z) such that

$$3^x + 11^y = z^2$$

2 Two nonperpendicular lines throught the point A and a point F on one of these lines different from A are given. Let P_G be the intersection point of tangent lines at G and F to the circle through the point A, F and G where G is a point on the given line different from the line FA. What is the locus of P_G as G varies.

3 We wish to color the cells of a $n \times n$ chessboard with k different colors such that for every $i \in \{1, 2, ..., n\}$, the $2n - 1$ cells on i. row and i. column have all different colors.

a) Prove that for $n = 2001$ and $k = 4001$, such coloring is not possible.

b) Show that for $n = 2^m - 1$ and $k = 2^{m+1} - 1$, such coloring is possible.