Saint Petersburg Mathematical Olympiad 2011

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–	**Grade 11**
1	$f(x), g(x)$ - two square trinomials and a, b, c, d - some reals. $f(a)=2, f(b)=3, f(c)=7, f(d)=10$ and $g(a)=16, g(b)=15, g(c)=11$ Find $g(d)$
2	ABC-triangle with circumcenter O and $\angle B=30$. BO intersect AC at K. L - midpoint of arc OC of circumcircle KOC, that does not contains K. Prove, that A, B, L, K are concyclic.
3	Can we build parallelepiped $6\times 7\times 7$ from $1\times 1\times 2$ bricks, such that we have same amount bricks of one of 3 directions ?
4	Call integer number x as far from squares and cubes, if for every integer k it is true : $\|x-k^2\|>10^6, \|x-k^3\|>10^6$. Prove, that there are infinitely many far from squares and cubes degrees of 2
5	Let $M(n)$ and $m(n)$ are maximal and minimal proper divisors of n Natural number $n>1000$ is on the board. Every minute we replace our number with $n+M(n)-m(n)$. If we get prime, then process is stopped. Prove that after some moves we will get number, that is not divisible by 17
6	$ABCD$ - convex quadrilateral. M -midpoint AC and $\angle MCB=\angle CMD=\angle MBA=\angle MBC-\angle MDC$. Prove, that $AD=DC+AB$
7	There is secret society with 2011 members. Every member has bank account with integer balance (can be negative). Sometimes some member give one dollar to every his friend. It is known, that after some such moves members can redistribute their money arbitrarily. Prove, that there are exactly 2010 pairs of friends.
–	**Grade 10**
1	Same as Grade11 P1
2	n - some natural. We write on the board all such numbers d, that $d\leq 1000$ and $d\|n+k$ for some $1\leq k\leq 1000$. Let $S(n)$ -sum of all written numbers. Prove , that $S(n)<10^6$ and $S(n)>10^6$ has infinitely many solutions.
3	Same as Grade11 P2

4 In some city there are 2000000 citizens. In every group of 2000 citizens there are 3 pairwise friends. Prove, that there are 4 pairwise friends in city.

5 Same as Grade11 P4

6 We have garland with n lights. Some lights are on, some are off. In one move we can take some turned on light (only turned on) and turn off it and also change state of neigbour lights. We want to turn off all lights after some moves.. For what n is it always possible?

7 $ABCD$ - convex quadrilateral. P is such point on AC and inside $\triangle ABD$, that

$$\angle ACD+\angle BDP=\angle ACB+\angle DBP=90-\angle BAD$$

.
Prove that $\angle BAD+\angle BCD=90$ or $\angle BDA+\angle CAB=90$

– **Grade 9**

2 a,b are naturals and

$$a\times GCD(a,b)+b\times LCM(a,b)<2.5ab$$

. Prove that $b|a$

3 Point D is inside $\triangle ABC$ and $AD=DC$. BD intersect AC in E. $\frac{BD}{BE}=\frac{AE}{EC}$. Prove, that $BE=BC$

4 Same as Grade10 P4

5 $ABCD$ - convex quadrilateral. $\angle A+\angle D=150, \angle B<150, \angle C<150$ Prove, that area $ABCD$ is greater than $\frac{1}{4}(AB*CD+AB*BC+BC*CD)$

6 There is infinite sequence of composite numbers $a_1,a_2,...,$ where $a_{n+1}=a_n-p_n+\frac{a_n}{p_n}$; p_n is smallest prime divisor of a_n. It is known, that $37|a_n$ for every n.
Find possible values of a_1

7 Sasha and Serg plays next game with 100-angled regular polygon . In the beggining Sasha set natural numbers in every angle. Then they make turn by turn, first turn is made by Serg. Serg turn is to take two opposite angles and add 1 to its numbers. Sasha turn is to take two neigbour angles and add 1 to its numbers. Serg want to maximize amount of odd numbers. What maximal number of odd numbers can he get no matter how Sasha plays?