National Olympiad Second Round 2003

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Day 1

1 $n \geq 2$ cars are participating in a rally. The cars leave the start line at different times and arrive at the finish line at different times. During the entire rally each car takes over any other car at most once , the number of cars taken over by each car is different and each car is taken over by the same number of cars. Find all possible values of n

2 Let $ABCD$ be a convex quadrilateral and K, L, M, N be points on $[AB], [BC], [CD], [DA]$, respectively. Show that,

$$\sqrt[3]{s_1} + \sqrt[3]{s_2} + \sqrt[3]{s_3} + \sqrt[3]{s_4} \leq 2\sqrt[3]{s}$$

where $s_1 = $ Area(AKN), $s_2 = $ Area(BKL), $s_3 = $ Area(CLM), $s_4 = $ Area(DMN) and $s = $ Area$(ABCD)$.

3 Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that
$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$

for all $x_1, x_2 \in \mathbb{R}$ and $t \in (0, 1)$. Show that
$\sum_{k=1}^{2003} f(a_{k+1})a_k \geq \sum_{k=1}^{2003} f(a_k)a_{k+1}$

for all real numbers $a_1, a_2, ..., a_{2004}$ such that $a_1 \geq a_2 \geq ... \geq a_{2003}$ and $a_{2004} = a_1$

Day 2

1 Suppose that $2^{2n+1} + 2^n + 1 = x^k$, where $k \geq 2$ and n are positive integers. Find all possible values of n.

2 A circle which is tangent to the sides $[AB]$ and $[BC]$ of $\triangle ABC$ is also tangent to its circumcircle at the point T. If I is the incenter of $\triangle ABC$, show that $\widehat{ATI} = \widehat{CTI}$

3 An assignment of either a 0 or a 1 to each unit square of an mxn chessboard is called $fair$ if the total numbers of 0s and 1s are equal. A real number a is called $beautiful$ if there are positive integers m, n and a fair assignment for the mxn chessboard such that for each of the m rows and n columns , the percentage of 1s on that row or column is not less than a or greater than $100 - a$. Find the largest beautiful number.