National Olympiad Second Round 2004

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1 In a triangle $\triangle ABC$ with$\angle B > \angle C$, the altitude, the angle bisector, and the median from A intersect BC at H, L and D, respectively. Show that $\angle HAL = \angle DAL$ if and only if $\angle BAC = 90^\circ$.

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2 Two-way flights are operated between 80 cities in such a way that each city is connected to at least 7 other cities by a direct flight and any two cities are connected by a finite sequence of flights. Find the smallest k such that for any such arrangement of flights it is possible to travel from any city to any other city by a sequence of at most k flights.

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3 **(a)** Determine if exist an integer n such that $n^2 - k$ has exactly 10 positive divisors for each $k = 1, 2, 3.$

(b) Show that the number of positive divisors of $n^2 - 4$ is not 10 for any integer $n.$

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4 Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying the condition $f(n) - f(n + f(m)) = m$ for all $m, n \in \mathbb{Z}$

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5 The excircle of a triangle ABC corresponding to A touches the lines BC, CA, AB at A_1, B_1, C_1, respectively. The excircle corresponding to B touches BC, CA, AB at A_2, B_2, C_2, and the excircle corresponding to C touches BC, CA, AB at A_3, B_3, C_3, respectively. Find the maximum possible value of the ratio of the sum of the perimeters of $\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$ and $\triangle A_3B_3C_3$ to the circumradius of $\triangle ABC$.

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6 Define $K(n, 0) = \varnothing$ and, for all nonnegative integers m and n, $K(n, m+1) = \{k|\ 1 \le k \le n$ and $K(k, m) \cap K$
Find the number of elements of $K(2004, 2004)$.
