National Olympiad Second Round 2005

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by WakeUp

1 For all positive real numbers a, b, c, d prove the inequality

$$\sqrt{a^4+c^4}+\sqrt{a^4+d^4}+\sqrt{b^4+c^4}+\sqrt{b^4+d^4}\geq 2\sqrt{2}(ad+bc)$$

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2 In a triangle ABC with $AB<AC<BC$, the perpendicular bisectors of AC and BC intersect BC and AC at K and L, respectively. Let O, O_1, and O_2 be the circumcentres of triangles ABC, CKL, and OAB, respectively. Prove that OCO_1O_2 is a parallelogram.

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3 Some of the $n+1$ cities in a country (including the capital city) are connected by one-way or two-way airlines. No two cities are connected by both a one-way airline and a two-way airline, but there may be more than one two-way airline between two cities. If d_A denotes the number of airlines from a city A, then $d_A\leq n$ for any city A other than the capital city and $d_A+d_B\leq n$ for any two cities A and B other than the capital city which are not connected by a two-way airline. Every airline has a return, possibly consisting of several connected flights. Find the largest possible number of two-way airlines and all configurations of airlines for which this largest number is attained.

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4 Find all triples of nonnegative integers (m,n,k) satisfying $5^m+7^n=k^3$.

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5 If a, b, c are the sides of a triangle and r the inradius of the triangle, prove that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\leq\frac{1}{4r^2}$$

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6 Suppose that a sequence $(a_n)_{n=1}^{\infty}$ of integers has the following property: For all n large enough (i.e. $n\geq N$ for some N), a_n equals the number of indices i, $1\leq i<n$, such that $a_i+i\geq n$. Find the maximum possible number of integers which occur infinitely many times in the sequence.
