National Olympiad Second Round 2006

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by Umut Varolgunes

Day 1

1 Points P and Q on side AB of a convex quadrilateral $ABCD$ are given such that $AP = BQ$. The circumcircles of triangles APD and BQD meet again at K and those of APC and BQC meet again at L. Show that the points D, C, K, L lie on a circle.

2 There are 2006 students and 14 teachers in a school. Each student knows at least one teacher (knowing is a symmetric relation). Suppose that, for each pair of a student and a teacher who know each other, the ratio of the number of the students whom the teacher knows to that of the teachers whom the student knows is at least t. Find the maximum possible value of t.

3 Find all positive integers n for which all coefficients of polynomial $P(x)$ are divisible by 7, where

$$P(x) = (x^2 + x + 1)^n - (x^2 + 1)^n - (x + 1)^n - (x^2 + x)^n + x^{2n} + x^n + 1.$$

Day 2

1 $x_1, ..., x_n$ are positive reals such that their sum and their squares' sum are equal to t. Prove that $\sum_{i \neq j} \frac{x_i}{x_j} \geq \frac{(n-1)^2 \cdot t}{t-1}$

2 ABC be a triangle. Its incircle touches the sides CB, AC, AB respectively at N_A, N_B, N_C. The orthic triangle of ABC is $H_A H_B H_C$ with H_A, H_B, H_C are respectively on BC, AC, AB. The incenter of $AH_C H_B$ is I_A; I_B and I_C were defined similarly.
Prove that the hexagon $I_A N_B I_C N_A I_B N_C$ has all sides equal.

3 Find all the triangles such that its side lenghts, area and its angles' measures (in degrees) are rational.