National Olympiad Second Round 2007

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Day 1 December 8th

1 In an acute triangle ABC, the circle with diameter AC intersects AB and AC at K and L different from A and C respectively. The circumcircle of ABC intersects the line CK at the point F different from C and the line AL at the point D different from A. A point E is choosen on the smaller arc of AC of the circumcircle of ABC . Let N be the intersection of the lines BE and AC . If $AF^2+BD^2+CE^2=AE^2+CD^2+BF^2$ prove that $\angle KNB=\angle BNL$.

2 Some unit squares of 2007×2007 square board are colored. Let (i,j) be a unit square belonging to the ith line and jth column and $S_{i,j}$ be the set of all colored unit squares (x,y) satisfying $x\le i, y\le j$. At the first step in each colored unit square (i,j) we write the number of colored unit squares in $S_{i,j}$. In each step, in each colored unit square (i,j) we write the sum of all numbers written in $S_{i,j}$ in the previous step. Prove that after finite number of steps, all numbers in the colored unit squares will be odd.

3 If a,b,c are three positive real numbers such that $a+b+c=3$, prove that $\frac{a^2+3b^2}{ab^2(4-ab)}+\frac{b^2+3c^2}{bc^2(4-ab)}+\frac{c^2+3a^2}{ca^2(4-ca)}\geq 4$

Day 2 December 9th

1 Let $k>1$ be an integer, $p=6k+1$ be a prime number and $m=2^p-1$.

Prove that $\frac{2^{m-1}-1}{127m}$ is an integer.

2 Let ABC be a triangle with $\angle B=90$. The incircle of ABC touches the side BC at D. The incenters of triangles ABD and ADC are X and Z , respectively. The lines XZ and AD are intersecting at the point K. XZ and circumcircle of ABC are intersecting at U and V. Let M be the midpoint of line segment $[UV]$. AD intersects the circumcircle of ABC at Y other than A. Prove that $|CY|=2|MK|$.

3 In a country between each pair of cities there is at most one direct road. There is a connection (using one or more roads) between any two cities even after the elimination of any given city and all roads incident to this city. We say that the city A can be *k-directionally* connected to the city B, if : we can orient at most k roads such that after *arbitrary* orientation of remaining roads for any fixed road l (directly connecting two cities) there is a path passing through roads in the direction of their orientation starting at A, passing through l and ending at B and visiting each city at most once. Suppose that in a country with n cities, any two cities can be *k - directionally* connected. What is the minimal value of k?