National Olympiad Second Round 2008

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Day 1

1 Given an acute angled triangle ABC , O is the circumcenter and H is the orthocenter.Let A_1,B_1,C_1 be the midpoints of the sides BC,AC and AB respectively. Rays $[HA_1$,$[HB_1$,$[HC_1$ cut the circumcircle of ABC at A_0,B_0 and C_0 respectively.Prove that O,H and H_0 are collinear if H_0 is the orthocenter of $A_0B_0C_0$

2 $a-)$ Find all prime p such that $\dfrac{7^{p-1}-1}{p}$ is a perfect square

$b-)$ Find all prime p such that $\dfrac{11^{p-1}-1}{p}$ is a perfect square

3 Let a.b.c be positive reals such that their sum is 1. Prove that $\frac{a^2b^2}{c^3(a^2-ab+b^2)}+\frac{b^2c^2}{a^3(b^2-bc+c^2)}+\frac{a^2c^2}{b^3(a^2-ac+c^2)}\geq \frac{3}{ab+bc+ac}$

Day 2

1 $f:\mathbb N\times \mathbb Z\rightarrow \mathbb Z$ satisfy the given conditions

$a)$ $f(0,0)=1$, $f(0,1)=1$,

$b)$ $\forall k \notin \left\{0,1\right\}$ $f(0,k)=0$ and

$c)$ $\forall n \geq 1$ and k , $f(n,k)=f(n-1,k)+f(n-1,k-2n)$

find the sum $\displaystyle\sum_{k=0}^{\binom{2009}{2}}f(2008,k)$

2 A circle Γ and a line ℓ is given in a plane such that ℓ doesn't cut Γ.Determine the intersection set of the circles has $[AB]$ as diameter for all pairs of $\left\{A,B\right\}$ (lie on ℓ) and satisfy $P,Q,R,S \in \Gamma$ such that $PQ \cap RS=\left\{A\right\}$ and $PS \cap QR=\left\{B\right\}$

3 There is a connected network with 2008 computers, in which any of the two cycles don't have any common vertex. A hacker and a administrator are playing a game in this network. On the $1st$ move hacker selects one computer and hacks it, on the $2nd$ move administrator selects another computer and protects it. Then on every $2k+1th$ move hacker hacks one more computer(if he can) which wasn't protected by the administrator and is directly connected (with an edge) to a computer which was hacked by the hacker before and on every $2k+2th$ move administrator protects one more computer(if he can) which wasn't hacked by the hacker and is directly connected (with an edge) to a computer which was protected by the administrator before for every $k > 0$. If both of them can't make move, the game ends. Determine the maximum number of computers which the hacker can guarantee to hack at the end of the game.