National Olympiad Second Round 2010

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Day 1

1 In a country, there are some two-way roads between the cities. There are 2010 roads connected to the capital city. For all cities different from the capital city, there are less than 2010 roads connected to that city. For two cities, if there are the same number of roads connected to these cities, then this number is even. k roads connected to the capital city will be deleted. It is wanted that whatever the road network is, if we can reach from one city to another at the beginning, then we can reach after the deleting process also. Find the maximum value of k.

2 Let P be an interior point of the triangle ABC which is not on the median belonging to BC and satisfying $\angle CAP = \angle BCP$. $BP \cap CA = \{B'\}$, $CP \cap AB = \{C'\}$ and Q is the second point of intersection of AP and the circumcircle of ABC. $B'Q$ intersects CC' at R and $B'Q$ intersects the line through P parallel to AC at S. Let T be the point of intersection of lines $B'C'$ and QB and T be on the other side of AB with respect to C. Prove that

$$\angle BAT = \angle BB'Q \iff |SQ| = |RB'|$$

3 Prove that for all $n \in \mathbb{Z}^+$ and for all positive real numbers satisfying $a_1a_2...a_n = 1$

$$\sum_{i=1}^{n} \frac{a_i}{\sqrt{{a_i}^4+3}} \le \frac{1}{2}\sum_{i=1}^{n}\frac{1}{a_i}$$

Day 2

1 Let A and B be two points on the circle with diameter $[CD]$ and on the different sides of the line CD. A circle Γ passing through C and D intersects $[AC]$ different from the endpoints at E and intersects BC at F. The line tangent to Γ at E intersects BC at P and Q is a point on the circumcircle of the triangle CEP different from E and satisfying $|QP| = |EP|$. $AB \cap EF = \{R\}$ and S is the midpoint of $[EQ]$. Prove that DR is parallel to PS.

2 For integers a and b with $0 \le a, b < 2010^{18}$ let S be the set of all polynomials in the form of $P(x) = ax^2 + bx$. For a polynomial P in S, if for all integers n with $0 \le n < 2010^{18}$ there exists a polynomial Q in S satisfying $Q(P(n)) \equiv n \pmod{2010^{18}}$, then we call P as a *good polynomial.* Find the number of *good polynomials.*

3 Let K be the set of all sides and diagonals of a convex $2010-gon$ in the plane. For a subset A of $K,$ if every pair of line segments belonging to A intersect, then we call A as an *intersecting set.* Find the maximum possible number of elements of union of two *intersecting sets.*
