National Olympiad Second Round 2012

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Day 1

1 Find all polynomials with integer coefficients such that for all positive integers n satisfies $P(n!) = |P(n)|!$

2 Let ABC be a isosceles triangle with $AB = AC$ an D be the foot of perpendicular of A. P be an interior point of triangle ADC such that $m(APB) > 90$ and $m(PBD) + m(PAD) = m(PCB)$. CP and AD intersects at Q, BP and AD intersects at R. Let T be a point on $[AB]$ and S be a point on $[AP$ and not belongs to $[AP]$ satisfying $m(TRB) = m(DQC)$ and $m(PSR) = 2m(PAR)$. Show that $RS = RT$

3 Find all non-decreasing functions from real numbers to itself such that for all real numbers x, y $f(f(x^2) + y + f(y)) = x^2 + 2f(y)$ holds.

Day 2

4 For all positive real numbers x, y, z, show that $\frac{x(2x-y)}{y(2z+x)} + \frac{y(2y-z)}{z(2x+y)} + \frac{z(2z-x)}{x(2y+z)} \geq 1$ is true.

5 Let P be the set of all 2012 tuples $(x_1, x_2, \dots, x_{2012})$, where $x_i \in \{1, 2, \dots 20\}$ for each $1 \leq i \leq 2012$. The set $A \subset P$ is said to be decreasing if for each $(x_1, x_2, \dots, x_{2012}) \in A$ any $(y_1, y_2, \dots, y_{2012})$ satisfying $y_i \leq x_i (1 \leq i \leq 2012)$ also belongs to A. The set $B \subset P$ is said to be increasing if for each $(x_1, x_2, \dots, x_{2012}) \in B$ any $(y_1, y_2, \dots, y_{2012})$ satisfying $y_i \geq x_i (1 \leq i \leq 2012)$ also belongs to B. Find the maximum possible value of $f(A, B) = \dfrac{|A \cap B|}{|A| \cdot |B|}$, where A and B are nonempty decreasing and increasing sets ($|\cdot|$ denotes the number of elements of the set).

6 Let B and D be points on segments $[AE]$ and $[AF]$ respectively. Excircles of triangles ABF and ADE touching sides BF and DE is the same, and its center is I. BF and DE intersects at C. Let $P_1, P_2, P_3, P_4, Q_1, Q_2, Q_3, Q_4$ be the circumcenters of triangles $IAB, IBC, ICD, IDA, IAE, IEC, ICF,$ respectively.

a) Show that points P_1, P_2, P_3, P_4 concylic and points Q_1, Q_2, Q_3, Q_4 concylic.
b) Denote centers of theese circles as O_1 and O_2. Prove that O_1, O_2 and I are collinear.