National Olympiad Second Round 2013

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Day 1 November 23rd

1 The circle ω_1 with diameter $[AB]$ and the circle ω_2 with center A intersects at points C and D. Let E be a point on the circle ω_2, which is outside ω_1 and at the same side as C with respect to the line AB. Let the second point of intersection of the line BE with ω_2 be F. For a point K on the circle ω_1 which is on the same side as A with respect to the diameter of ω_1 passing through C we have $2 \cdot CK \cdot AC = CE \cdot AB$. Let the second point of intersection of the line KF with ω_1 be L. Show that the symmetric of the point D with respect to the line BE is on the circumcircle of the triangle LFC.

2 Let m be a positive integer.
a. Show that there exist infinitely many positive integers k such that $1 + km^3$ is a perfect cube and $1 + kn^3$ is not a perfect cube for all positive integers $n < m$.
b. Let $m = p^r$ where $p \equiv 2 \pmod 3$ is a prime number and r is a positive integer. Find all numbers k satisfying the condition in part a.

3 Let G be a simple, undirected, connected graph with 100 vertices and 2013 edges. It is given that there exist two vertices A and B such that it is not possible to reach A from B using one or two edges. We color all edges using n colors, such that for all pairs of vertices, there exists a way connecting them with a single color. Find the maximum value of n.

Day 2 November 24th

1 Find all positive integers m and n satisfying $2^n + n = m!$.

2 Find the maximum value of M for which for all positive real numbers a, b, c we have

$$a^3 + b^3 + c^3 - 3abc \geq M(ab^2 + bc^2 + ca^2 - 3abc)$$

3 Let n be a positive integer and $P_1, P_2, \ldots, P_n$ be different points on the plane such that distances between them are all integers. Furthermore, we know that the distances $P_iP_1, P_iP_2, \ldots, P_iP_n$ forms the same sequence for all $i = 1, 2, \ldots, n$ when these numbers are arranged in a non-decreasing order. Find all possible values of n.