Turkey Team Selection Test 1990

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1 The circles k_1,k_2,k_3 with radii $(a>c>b)$ a,b,c are tangent to line d at A,B,C, respectively. k_1 is tangent to k_2, and k_2 is tangent to k_3. The tangent line to k_3 at E is parallel to d, and it meets k_1 at D. The line perpendicular to d at A meets line EB at F. Prove that $AD=AF$.

2 For real numbers x_i, the statement

$$x_1+x_2+x_3=0 \Rightarrow x_1x_2+x_2x_3+x_3x_1 \leq 0$$

is always true. (Prove!)
For which $n\geq 4$ integers, the statement

$$x_1+x_2+\dots+x_n=0 \Rightarrow x_1x_2+x_2x_3+\dots+x_{n-1}x_n+x_nx_1 \leq 0$$

is always true. Justify your answer.

3 Let n be an odd integer greater than 11; $k\in \mathbb{N}$, $k \geq 6$, $n=2k-1$.
We define

$$d(x,y) = \left | \left \{ i\in \{1,2, \dots, n \} \bigm | x_i \neq y_i \right \} \right |$$

for $T=\{ (x_1, x_2, \dots, x_n) \bigm | x_i \in \{0,1\}, i=1,2, \dots, n \}$ and $x=(x_1,x_2,\dots,x_n), y=(y_1, y_2, \dots, y_n) \in T$.
Show that $n=23$ if T has a subset S satisfying

-$|S|=2^k$
-For each $x \in T$, there exists exacly one $y\in S$ such that $d(x,y)\leq 3$

4 Let $ABCD$ be a convex quadrilateral such that

$$\begin{matrix} E,F\in [AB], & AE=EF=FB \\ G,H\in [BC], & BG=GH=HC \\ K,L\in [CD], & CK=KL=LD \\ M,N\in [DA], & DM=MN=NA \end{matrix}$$

Let

$$[NG]\cap [LE]=\{P\}, [NG]\cap[KF]=\{Q\},$$
$$[MH]\cap[KF]=\{R\}, [MH]\cap[LE]=\{S\}$$

Prove that -$Area(ABCD)=9\cdot Area(PQRS)$ - $NP=PQ=QG$

5 Let b_m be numbers of factors 2 of the number $m!$ (that is, $2^{b_m}|m!$ and $2^{b_m+1} \nmid m!$). Find the least m such that $m - b_m = 1990$.

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6 Let $k \geq 2$ and $n_1, \dots, n_k \in \mathbf{Z}^+$. If $n_2|(2^{n_1} - 1)$, $n_3|(2^{n_2} - 1)$, $\dots$, $n_k|(2^{n_{k-1}} - 1)$, $n_1|(2^{n_k} - 1)$, show that $n_1 = \dots = n_k = 1$.
