**Turkey Team Selection Test 1991**

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**Day 1**

**1** Let $C', B', A'$ be points respectively on sides $AB, AC, BC$ of $\triangle ABC$ satisfying $\frac{AB'}{B'C} = \frac{BC'}{C'A} = \frac{CA'}{A'B} = k$. Prove that the ratio of the area of the triangle formed by the lines $AA', BB', CC'$ over the area of $\triangle ABC$ is $\frac{(k-1)^2}{(k^2+k+1)}$.

**2** Show that the equation $a^2 + b^2 + c^2 + d^2 = a^2 \cdot b^2 \cdot c^2 \cdot d^2$ has no solution in positive integers.

**3** Let $f$ be a function on defined on $|x| < 1$ such that $f\left(\frac{1}{10}\right)$ is rational and $f(x) = \sum_{i=1}^{\infty} a_i x^i$ where $a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Prove that $f$ can be written as $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials with integer coefficients.

**Day 2**

**1** A frog is jumping on $N$ stones which are numbered from $1$ to $N$ from left to right. The frog is jumping to the previous stone (to the left) with probability $p$ and is jumping to the next stone (to the right) with probability $1 - p$. If the frog has jumped to the left from the leftmost stone or to the right from the rightmost stone, it will fall into the water. The frog is initially on the leftmost stone. If $p < \frac{1}{3}$, show that the frog will fall into the water from the rightmost stone with a probability higher than $\frac{1}{2}$.

**2** $p$ passengers get on a train with $n$ wagons. Find the probability of being at least one passenger at each wagon.

**3** Let $U$ be the sum of lengths of sides of a tetrahedron (triangular pyramid) with vertices $O, A, B, C$. Let $V$ be the volume of the convex shape whose vertices are the midpoints of the sides of the tetrahedron. Show that $V \leq \frac{(U-|OA|-|BC|)(U-|OB|-|AC|)(U-|OC|-|AB|)}{(2^7 \cdot 3)}$.